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MEMOIR

RESPECTING

A NEW THEORY OF NUMBERS.

MEMOIR

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A NEW THEORY OF NUMBERS.

PART FIRST.

By CHARLES BROUGHTON, Esq.
EDINBURGH.

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ERRATA.

Page 30, line 10, *for* $\frac{9}{49}$, *read* $\frac{2}{49}$.

ib. 12, *for* 25,000 *read* 25,200.

ib. 2d from the bottom, *for* $\overset{Tx-1}{3}$ *read* $\overset{Tx-1}{3}$

31 5 from ditto, *for* $=34+2$ *read* and $34+2$.

32 6 and 7, for $209\times 2\doteq 418+2=3x=420=3x=$ and x 1.40, *read* $2.09\times 2=3x-2$ and $4.18+2=3x=420$, and $x=1.40$.

40 7 from the bottom, *for* $\frac{2}{2}\left(\overset{Tx-1}{3}\right)1$ *read* $\frac{2}{2}\left(\overset{Tx-1}{3}\right)$.

47 13 *for* $\overset{Tx-1}{5}+\overset{Tx}{5}$ *read* $2\left(\overset{Tx-1}{5}\right)+\overset{Tx}{4}$.

ib. ib. *for* 30 *read* 20.

54 18, *for* 48 *read* 84.

67 3, bottom lines, *for* $\frac{\overset{9}{7}}{6}$ *read* $\frac{9}{\overset{7}{6}}$.

78 15, *for* $1.36=m$, *read* $1.36, m$.

79 bottom line, *for* $-x^3$ *read* or x^3.

82 6, *for* $5-x=$ *read* $5=x$.

83 1st and 3d from bottom, leave out $=0$.

96 2d from bottom, *for* $=2x^3-3x$ *read* $=2x^3-x$.

There are several other errors of the press, which, as they are easily corrected, it has not been thought necessary to notice.

MEMOIR

RESPECTING

A NEW THEORY OF NUMBERS.

INTRODUCTION.

I HAPPENED, about two years ago, to be engaged in the investigation of a problem depending upon a series of ternary combinations, when I observed, that cube numbers are multiples of 6, with the addition of their roots; that the cube of 2, for instance, is $6 + 2 = 8$; and the cube of 3 is $27 = 24 + 3$, that is, $6 \times 4 + 3$, the cube of $4 = 64 = 60 + 4$, or $6 \times 10 + 4$, &c. &c.

But although this discovery (as I then imagined it to be,) excited my curiosity, yet I did not think of carrying my researches farther until last November, that, looking over some old papers, this forgotten formation of cubes fell again under my observation, and I was induced to consider it with attention.

I found that the series of factors of 6, composing, with the addition of the roots, the series of cubes, obeyed this law, 1, $1 + 3$, $1 + 3 + 6$, $1 + 3 + 6 + 10$, $1 + 3 + 6 + 10 + 15$, &c. being a series formed of the successive terms of the series of trigonal numbers. And I likewise observed, that the series of cubes themselves formed, in their turn, the se-

ries of squares of the trigonal numbers; thus 1, $1 + 8 = 9 = 3^2$, $1 + 8 + 27 = 36 = 6^2$, $1 + 8 + 27 + 64 = 100 = 10^2$, &c. being the series 1^2, 3^2, 6^2, 10^2, &c.

This evident connection between the cubes, or 3d powers of numbers, and the series of trigonal numbers, or 3d order of numbers, * attracted very forcibly my attention; and led me to investigate also the relation between the other powers of numbers, and the corresponding orders of numbers. This I did alternately by synthesis and analysis; *synthetically*, by raising a regular series of terms of the series of units, composing the series of the natural numbers, to the same power, and multiplying continually by the series of units in the root—and *analytically*, by resolving a series of the natural numbers, when raised to the same power, into their successive and ultimate differences. And I had the pleasure to observe throughout, a constant and invariable relation between the several *powers* and the corresponding *orders* of numbers. And, after much labour, I succeeded ultimately in obtaining, both by synthesis and analysis, general formulæ in terms of the orders of numbers, for the resolution of equations of the first ten degrees, to which I had limited my inquiries.

When engaged in these investigations, I was much struck with the beautiful order and harmony which everywhere obtained in the combinations of the powers of numbers, and particularly as proceeding in a regular progression from a first series of units; and it was a natural step, in this train of thought, to consider these units in the abstract, as mere mathematical points; and, placing them at regular, fixed, and equal distances, to consider them as representing lineal space and time: That unity itself was without magnitude, and could not be in-

* I have explained, in a preliminary chapter, the trigonal numbers and the other orders of numbers.

creased or diminished by multiplication or division by itself, but was unchangeable through all its powers; whereas, two points or more could represent a line, which, being multiplied into itself, would denote a line carried laterally in the same plane, and so represent a geometrical square; and that this square, multiplied again by the points in the root representing the root or line, now placed vertically above it, would form a cube or solid, corresponding to the number of points in the root or line. But as *two* points were the least that could represent lineal space, so *three* were required to denote superficial space, and *four* to form solid or cubical space,—the most regular position of such points being at equal distances, or in the form of an equilateral triangle and a pyramid. This led almost immediately again to the summation of the triangular numbers, as representing triangular space, and consequently, to finding the number of points, or numeral expression, for all the regular polygons, and ultimately to a new and very convenient quadrature of the circle and of the cone.

But, after I had proceeded thus far, and found the geometrical construction of the powers of numbers exceeding three degrees, by substituting their lower powers for a line or a square, so as to confine them within three dimensions, (by which I found that every pure equation might be reduced to an equivalent cubic equation)—I again returned to the more ordinary consideration of number as representing quantity in general, independent of *position*, its necessary condition in geometry. And after having completed my theorems for the resolution of equations, by means of the orders of numbers, I prepared for their immediate publication, with tables of the orders of numbers, and rules for practice.

The publication, however, was again interrupted by a new object of investigation which suggested itself—the nature of

prime numbers—and I was fortunate enough, not only to discover the law of their negative formation, if I may so express it, (for they are not found directly, but by excluding all other numbers which are products, or composed of factors,) but also to find a method of constructing tables of all the prime factors of numbers, without the necessity of calculation.

Most men, who have been much occupied in the pursuit of any object of research and difficulty, are apt to consider it, when obtained, with reference to the labour it has cost them, and so frequently to overrate its real value. This, I am afraid, has been my case. When I first discovered these general theorems for the resolution of the higher equations, and the method of finding the prime factors of numbers, without calculation and to any extent, I was hurried along by an idea of their importance to science, much above what I have found to be justified by any interest they have hitherto excited in some eminent persons, the best qualified to judge of their real utility; and, I have been advised by those at the very head of science in England, that there are now so few mathematical readers in this country, and so little chance of encouragement for any improvement in abstract science, that it would be extremely hazardous for me to publish a complete treatise, accompanied by tables, on these subjects; and, in conformity with this advice, I have limited the object of the present publication to a short memoir or syllabus of my methods, with no more detail than is necessary for understanding the theorems.

This I shall do under the following heads:

I.

Theorems for the resolution of equations of the first ten degrees, which I subdivide into,

1. Theorems in terms of the orders of numbers, derived from the synthesis of powers.
2. Particular theorems, also derived from the synthesis, and related to the orders of numbers.
3. Theorems in terms of the orders of numbers, derived from the analysis.
4. Miscellaneous theorems, for the more easy and expeditious resolutions of equations, and particularly of quadratic and cubic equations.
5. A General theorem for the resolution of equations, and the summation of series.
6. A general theorem for finding the series of terms of any order of numbers, m.

II.

Theorems respecting the summation of an infinite series of fractional powers.

III.

Method of constructing tables of the prime factors of numbers, and of the prime numbers.

IV.

Theorems for finding products and quotients, by addition and subtraction merely, with the method of constructing the tables for finding quotients and products.

V.

Theorems for the quadrature of the regular polygons, and of the circle.

PRELIMINARY CHAPTER.

OF THE POWERS OF NUMBERS, AND THE BINOMIAL THEOREM.

THE successive multiplications of a number into itself, form the powers of that number; and these powers are denominated according to the number of the successive multiplications. Thus, the first power of a number is the number itself, or the number taken once; and this first power of any number is called, with relation to the higher powers of that number, the root of these higher powers.

The second power of a number is formed by the multiplication of that number or root, by the number itself; thus 9 is the second power of 3, for $3 \times 3 = 9$, and 3 is also called the root, and 9 the square.

The third power is derived from the second power by multiplying again by the number itself, or root; thus 27 is the third power of 3, for $9 \times 3 = 27$. And as second powers are likewise termed squares, so third powers have also been denominated cubes, from the relation between the products of numbers considered as *lines*, and surfaces and solids in geometry.

The fourth power of a number is derived from the third power, multiplying again by the root; or it may be considered

as the square of a square, or the second power of the second power of the number: Thus 81 is either 27×3, the product of the third power multiplied by the root, or it is 9×9, the product of the square into the square. In the former case, it is properly the fourth power of 3. In the latter case, it is the second power of 9.

Fifth powers are in like manner derived from the fourth powers, multiplying again by the root; and so for all the higher powers, which are each in succession the product of the preceding power multiplied by the root, or by some other power of the number; so that, to obtain the higher powers of any number, it was necessary to obtain previously the lower powers of that number. This, besides being a very tedious and laborious process, was attended with this difficulty, that we could not easily retrace our steps, and from the *power*, find the *root* of that power. Thus, it is not apparent that 169 is the square or second power of 13, and still less, that 2197 is its cube or third power; and the higher the power, our difficulty in finding the root is increased.

The invention of logarithms obviated this difficulty in the involution and evolution of the *perfect* powers of numbers. As the power of any number was obtained by the mere multiplication of the logarithm of the number by the index of the power, and *vice versa*, the logarithm of the root was obtained by dividing the logarithm of the power by its index.

But where the powers of numbers were not perfect, but combined in any way with other powers of the same number, or with the root itself, the question could not be resolved by logarithms. And, indeed, where these other powers or terms were also affected by other factors, it has not hitherto been found practicable to resolve the equations of the higher degrees, otherwise than by infinite series.

In the investigation of the roots of numeral powers, special arithmetic has derived the demonstration of all her methods from algebra. Thus, the rule in arithmetic for finding the square root of a number is obtained from the algebraic formula, $(a+b)^2 = a^2 + 2ab + b^2$; for as all numbers may be considered as divisible into two or more parts, these parts may be denoted by letters, or any other general symbol, and the powers of that number will evidently be composed of the terms of its parts when raised to the power. Thus, if the number is supposed to be divided into two parts, and if these parts shall be represented by a and b, then will the powers of any number m be found in terms of the binomial $a+b$ raised to its successive powers, thus,

First power $= a + b = m$

multiplying by the root, $a + b$

$$a^2 + ab$$
$$+ ab + b^2$$

Second power, $(a+b)^2 = a^2 + 2ab + b^2 = m^2$

multiplying by the root, $+ a + b$

$$a^3 + 2a^2b + ab^2$$
$$+ a^2b + 2ab^2 + b^3$$

Third power, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = m^3$

multiplying by the root, $a + b$

$$a^4 + 3a^3b + 3a^2b^2 + ab^3$$
$$+ a^3b + 3a^2b^2 + 3ab^3 + b^4$$

Fourth power, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = m^4$

multiplying by the root, $a + b$

$$a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4$$
$$+ a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5$$

Fifth power, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = m^5$

&c.

And in *arithmetic*, by the application of these formulæ, with due attention to the decimal notation, it is possible, though very laborious, to find the roots of perfect powers, by finding successively all the figures of which the root is composed; beginning with the figure of the highest rank, or a, and then seeking b, the second figure; and having found b, taking their sum, or $a + b$, for the first term of the root, or a, and searching again for the second term, or b; and so constantly substituting the terms found for the first term, and repeating the operation until all the terms of the root are found. But as this is so much more easily accomplished by logarithms, these formulæ would have been of little use, had they not contributed to the invention of the celebrated Binomial Theorem of Sir Isaac Newton.

This general theorem for any binomial power, such as $a + b$ raised to the power m, is as follows,

$$(a+b)^m = a^m + \frac{m.}{1.}a^{m-1}b + \frac{m.m-1}{1.2}a^{m-2}b^2 + \frac{m.m-1.m-2}{1.2.3}a^{m-3}b^3$$
$$+ \frac{m.m-1.m-2.m-3}{1.2.3.4}a^{m-4}b^4 + \frac{m.m-1.m-2.m-3.m-4}{1.2.3.4.5}a^{m-5}b + \text{&c.}$$

Sir Isaac Newton, in his letter to Mr Oldenburgh, explains, that he was led to the discovery of this theorem by investigating the laws of the interpolation of the series exhibited by Dr Wallis, in his Arithmetic of Infinites for finding the quadrature of the circle and hyperbola: And, amongst other things, he remarks, that in investigating the coefficients of the numerators of the interpolated powers, he found them to be "the same "with the figures of which the several powers of 11 consist, "viz. of 11^0, 11^1, 11^2, 11^3, &c. that is, the first 1

"the second 1, 1

"the third 1, 2, 1

" the fourth 1, 3, 3, 1
" the fifth 1, 4, 6, 4, 1, &c.

" I inquired, therefore, (says this great man,) how, in these " series, the rest of the terms may be derived from the first two " being given; and I found, that, by putting m for the second " figure or term, the rest would be produced by the continued " multiplication of the terms of this series, $\frac{m-0}{1} \times \frac{m-1}{2} \times$ " $\frac{m-3}{3} \times \frac{m-4}{4} + \frac{m-5}{5}$ &c.

" For instance, if the second term $m=4$, then shall $4 \times \frac{m-1}{2}$ " or 6 be the third term, and $6 \times \frac{m-2}{3}$, or 4, the fourth term, and " $4 \times \frac{m-3}{4}$, or 1, the fifth term, and $1 \times \frac{m-4}{5}$ or 0, the sixth, " which shews that in this case the series terminates."

He shews, in the same letter, how he discovered the application of this theorem to the extraction of roots, by making m a *fraction*, and calculating the successive terms of this fractional power, the terms of which will not terminate as in integral powers, but go on to an infinite series. But this is foreign to our present inquiry.

Had Sir Isaac Newton happened to observe the *summations* of these coefficients, as well as their value, according to the decimal notation of arithmetic, it is probable that he would have been led to consider them as the successive powers of the binomial $1 + 1$ or 2, in place of the figures of the powers of the binomial $10 + 1$ or 11. For the first power $1 = 2^0$; the second power, or the first which belongs to a binomial, is $1 + 1$, which summed, is $2 = 2^1$; the second binomial power is $1 + 2 + 1$, the sum of which is 4, or 2^2; the third binomial power

$1 + 3 + 3 + 1$ is 8, or 2^3; the fourth binomial power $1 + 4 + 6 + 4 + 1$ is 16, or 2^4; and so on for the other powers of this binomial. And the improvement of this observation could not have escaped his sagacity.

For the identity of the coefficients found by the binomial theorem for the successive powers of any binomial $a + b$, with the series of powers of the binomial $1 + 1$ or 2, naturally suggests to us to raise the binomial $1 + 1$ to its successive powers, from which we are likely to obtain some law applicable to the binomial theorem itself.

Thus the first power is $1+1=2=2^1$

multiplying by the root $1+1$

we obtain $1+1$

$1+1$

$(1+1)^2=1+2+1=4=2^2$

multiplying again by the root, $1+1$

$1, 2, 1$

$1, 2, 1$

$(1+1)^3 = 1, 3, 3, 1=8=2^3$

multiplying by the root, $1+1$

$1, 3, 3, 1$

$1, 3, 3, 1$

$(1+1)^4 = 1, 4, 6, 4, 1=16=2^4$

multiplying again by the root, $1+1$

$1, 4, 6, 4, 1$

$1, 4, 6, 4, 1$

$(1+1)^5 = 1, 5, 10, 10, 5, 1=32=2^5$

$(1+1)^5 = 1, 5, 10, 10, 5, 1 = 32 = 2^5$

multiplying again by the root, $1+1$

$1, 5, 10, 10, 5, 1$

$1, 5, 10, 10, 5, 1$

$(1+1)^6 = 1, 6, 15, 20, 15, 6, 1 = 64 = 2^6$

multiplying again by the root, $1+1$

$1, 6, 15, 20, 15, 6, 1$

$1, 6, 15, 20, 15, 6, 1$

$(1+1)^7 = 1, 7, 21, 35, 35, 21, 7, 1 = 128 = 2^7$

multiplying again by the root, $1+1$

$1, 7, 21, 35, 35, 21, 7, 1$

$1, 7, 21, 35, 35, 21, 7, 1$

$(1+1)^8 = 1, 8, 28, 56, 70, 56, 28, 8, 1 = 256 = 2^8$

multiplying again by the root, $1+1$

$1, 8, 28, 56, 70, 56, 28, 8, 1,$

$1, 8, 28, 56, 70, 56, 28, 8, 1$

$(1+1^9) = 1, 9, 36, 84, 126, 126, 84, 36, 9, 1 = 512 = 2^9$

multiplying again by the root, $1+1$

$1, 9, 36, 84, 126, 126, 84, 36, 9, 1$

$2, 9, 36, 84, 126, 126, 84, 36, 9, 1$

$(1+1)^{10} = 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1 = 1024$

$= 2^{10}$ &c.

Which terms agree with the law of the exponents indicated by the binomial theorem. Thus if $m=6$, the terms of the theorem,

$$1+\frac{m}{1}+\frac{m.m\text{-}1}{1\ .\ 2}+\frac{m\ .\ m\text{-}1\ .\ m\text{-}2}{1\ .\ 2\ .\ 3}+\frac{m.m\text{-}1\ .\ m\text{-}2.m\text{-}3}{1\ .\ 2\ .\ 3\ .\ 4}+$$

$$\frac{m.m\text{-}1.m\text{-}2.m\text{-}3.m\text{-}4}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5}+\frac{m.m\text{-}1.m\text{-}2.m\text{-}3.m\text{-}4.m\text{-}5}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6}$$ are equal to this series,

$$=1+\frac{6}{1}+\frac{6\times 5}{1\times 2}+\frac{6\times 5\times 4}{1\times 2\times 3}+\frac{6\times 5\times 4\times 3}{1\times 2\times 3\times 4}+$$

$$=1+6+15+20+15+$$

$$\frac{6.5.4.3.2}{1.2.3.4.5}+\frac{6.5.4.3.2.1}{1.2.3.4.5.6}$$

6 + 1 being

the terms we have found by raising the binomial 1 + 1 to the sixth power.

We may observe farther, in regard to these coefficients, that the first terms are all units; and that the second terms form the series of natural numbers, thus:

Second term of the 1st power,	1	&c. being the series of natural numbers.
2d,	2	
3d,	3	
4th,	4	
5th,	5	
6th,	6	

The third terms are as follows;

The third term of the 1st power,	0	being the series of trigonal numbers; or the series of the sums of the natural numbers.
2d,	1	
3d,	3	
4th,	6	
5th,	10	
6th,	15	

The coefficients of the fourth terms are,

For the 1st power	0	being the series formed by the continued summation of the trigonal numbers; thus $1+3=4$, $4+6=10$, $10+10=20$, $20+15=35$, &c.
2d,	0	
3d,	1	
4th,	4	
5th,	10	
6th,	20	

The coefficients of the fifth terms are,

For the 1st power,	0	being the series formed by the continued summation of the former series, 1, 4, 10, 20, &c.; thus 1 is 1, $1+4=5$, $5+10=15$, $15+20=35$, &c.
2d,	0	
3d,	0	
4th,	1	
5th,	5	
6th,	15	
7th,	35	

The coefficients of the sixth terms begin at the fifth powers, and are as under:

For the 5th power,	1	being the series formed by the continued summations of the former series, 1, 5, 15, 35, 70, &c.; thus, 1 is 1., $1+5=6$ $6+15=21$, $21+35=56$, $56+70=126$, &c.
6th,	6	
7th,	21	
8th,	56	
9th,	126	

The coefficients of the seventh terms obey the same law.

For the 6th power,	1	which will be found by summing the preceding series, 1, 6, 21, 56, 126, &c.; thus, 1 is 1, $1+6=7$, $7+21=28$, $28+56=84$, $84+126=210$.
7th,	7	
8th,	28	
9th,	84	
10th,	210	

The general law is therefore evident, for finding the different series of terms of a binomial $1+1$, raised successively through all its several powers, viz. that each succeeding series

is found by the continued summations of the former series. Thus, the series of first terms are all units; and, by summing such series continually, we obtain the following table:

TABLE OF BINOMIAL COEFFICIENTS and of the Orders of Numbers.

	1 Powers	2 Powers	3 Powers	4 Powers	5 Powers	6 Powers	7 Powers	8 Powers	9 Powers	
1 Order or Series	1	1	1	1	1	1	1	1	1	1 Terms
2 Order	1	2	3	4	5	6	7	8	9	10 2 Terms
3 Order	1	3	6	10	15	21	28	36	45	55 3 Terms
4 Order	1	4	10	20	35	56	84	120	165	220 4 Terms
5 Order	1	5	15	35	70	126	210	330	495	715 5 Terms
6 Order	1	6	21	56	126	252	462	792	1287	2002 6 Terms
7 Order	1	7	28	84	210	462	924	1716	3003	5005 7 Terms
8 Order	1	8	36	120	330	792	1716	3432	6435	11440 8 Terms
9 Order	1	9	45	165	495	1287	3003	6435	12870	24310 9 Terms
10 Order	1	10	55	220	715	2002	5005	11440	24310	48620 10 Terms
	1 Terms	2 Terms	3 Terms	4 Terms	5 Terms	6 Terms	7 Terms	8 Terms	9 Terms	10 Terms

(Coefficients of)

We observe, *first*, from this table, that the series of the coefficients of the same terms of a binomial, raised to its several successive powers, are the same with what shall be called the successive *orders of numbers*.

Thus all these binomial terms, *or orders of numbers*, proceed from a first series of units, and the series of units is therefore the *first order* of numbers. Thus,

First *series or order*,	1,	1,	1,	1,	1,	1,	1,
Their sums, *second order*,	1,	2,	3,	4,	5,	6,	7,
Their sums, *third order*,	1,	3,	6,	10,	15,	27,	28,
Their sums, *fourth order*,	1,	4,	10,	20,	35,	56,	84, &c.

The sums of the first order or series of units form the natural numbers, 1, 2, 3, 4, 5, 6, &c. which shall therefore be called numbers of the *second order*. The numbers commonly called trigonals, being the series of numbers obtained by the continued summation of the natural numbers, form the *third order* of numbers; the series produced by the summation of the numbers of the *third order*, form the *fourth order*, and so of the rest.

We observe, *secondly*, That the tabular series, when read from top to bottom, are the same, rank for rank, as when read from left to right; that is, the successive summations of the series of numbers are the same, rank for rank, with the series of the different successive terms of the several powers of a binomial.

Thirdly, If we read this table diagonally, beginning at the upper corner on the left, we shall find the coefficients of the binomial theorem, or the terms of the powers of the binomial $1 + 1$. Thus, the first figure is 1, being equal to $(1+1)^0$ or $\frac{1+1}{1+1} = 1$.

The figures of the 2d diagonal line are $1 + 1 = 2$, the binomial itself $= (1+1)^1$

The 3d diagonals are $1 + 2 + 1 = 4$, the binomial raised to the square.

The 4th diagonals are $1 + 3 + 3 + 1 = 8$, the binomial raised to the cube.

The 5th diagonals are $1 + 4 + 6 + 4 + 1 = 16$ the bino-

mial raised to the biquadrate or 4th power, and so for the higher powers, as far as the table is carried.

Farther, we observe that the series of terms of the different powers are reciprocally the same in respect to the coefficients: Thus, the 2*d* term of the 3*d* power has the same coefficient with the 3*d* term of the 2*d* power; the 2*d* term of the 4*th* power has the same coefficient with the 4*th* term of the 2*d* power; the 2*d* term of the 5*th* power has the like coefficient with the 5*th* term of the 2*d* power; the 3*d* term of the 4*th* power the same with the 4*th* term of the 3*d* power, and so on. The terms equally distant from the middle term being always equal, and the middle terms themselves, either equal to each other, if there are two middle terms, as happens when the index of the power is an *odd* number, or equal to the sum of the middle terms of the preceding powers, when there is but one middle term, as is always the case when the index of the power is *even*.

Although I was led in this way to the formation of the binomial table, or table of the *orders of numbers*, without being aware of its having been observed before, yet I have since found that it has been pointed out by others; but no other advantage seems to have been derived from it, than as it afforded an easy method of finding the binomial coefficients. We shall immediately perceive, however, that these orders of numbers are capable of a much more general and extensive application.

It has been already remarked, that all numbers are divisible into parts, which may be denoted by as many separate letters as the number of the parts into which the number itself is supposed to be divided; and it matters nothing, in this view, whether the given number shall be considered as an *integer*, or as a fraction; for the part of a number is itself a whole number, if viewed in relation to the lesser parts of which it is the multiple. After this manner of considering number, all num-

bers may be considered as integral, and the greatest number of integral parts, of which any number can consist, is the number of units, or least parts which it contains. * In order, therefore, to inquire into the relation of the powers of numbers to each other, it seemed to be the most natural method to begin the investigation with unity itself, and to proceed, step by step, and unit by unit, until we should obtain some general law of the powers of a series of numbers or roots, in arithmetical progression, which might be applied to all other numbers raised to the same power. For it was quite evident, that as the powers of any one number, taken by itself, could only shew that these powers are derived by multiplying the products constantly by the number itself, or the root; so by raising a binomial to its successive powers, we could only obtain the laws for binomial powers, or powers of two terms. But if we were to observe and compare the laws of the powers of binomials, of trinomials, of quadrinomials, &c. and so on, in a *series* or *succession*, it appeared at least possible, that we might observe some general law among the powers of the series of the progressions of polynomials themselves, whereby to attain to the knowledge of some truths which have been hitherto hid from us.

This accordingly was the object of a very tedious investigation by synthesis, of which I shall only state the results, in the following chapter.

* This expression is perhaps too broad in the text, as there are fractions which, expressed in decimals, do not terminate, and to the value of which, therefore, we can only approximate by series; but there is no limit to the degree of the approximation.

CHAPTER I.

THEOREMS FOR THE RESOLUTION OF EQUATIONS, IN TERMS OF THE ORDERS OF NUMBERS, DERIVED FROM THE SYNTHESIS OF POWERS.

In order to understand more clearly the relation between the orders of numbers and the powers of numbers, it will be necessary to consider the formation of the orders of numbers as arising from multiplication, as well as from the summations of series.

We have observed already, that all number is composed of units, which may be arranged in an infinite series, comprehending all integral numbers $1+1+1+1+1+1+1+$ &c. or without the signs, which will only perplex us; thus, 1, 1, 1, 1, 1, 1, 1, &c. which is the 1st order of the natural numbers or units. But we may have the double of this series, viz. 2, 2, 2, 2, 2, &c. which is the 1st order of 2; or we may assume any other multiple of 1 whatever, m, as m, m, m, m, m, m, &c. and the series will be the 1st order of m.

The 2d order of numbers is formed by the continued summations of the successive terms of the 1st order; thus, 1, $1+1$

$=2$, $2+1=3$, $3+1=4$, &c. the series of natural numbers 1, 2, 3, 4, 5, which series is the 2d order of 1. The 2d order of 2 being 2, $2+2=4$, $4+2=6$, $6+2=8$, the series of even numbers.

The odd numbers belong also to the same order, being the 2d order of 2, + 1st order of 1; thus,

2d order of 2, being	0, 2, 4, 6, 8, 10, 12, &c.
Adding 1st order of 1, we obtain the odd numbers	1, 3, 5, 7, 9, 11, 13, &c.

Generally the 2d order of m is m, $2m$, $3m$, $4m$, $5m$, &c.; and adding any number x, we have x, $m+x$, $2m+x$, $3m+x$, &c. = 2d order of m + 1st order of x.

It has been shewn, in the preceding chapter, that the 3d order of numbers is, in like manner, obtained by the continued summations of the terms of the 2d order: Thus the sums of the natural numbers 1, 2, 3, 4, 5, &c. form the series of the trigonal numbers 1, 3, 6, 10, &c.; or, generally, the series m, $3m$, $6m$, $10m$, $15m$ + &c. is made up of the successive sums of the terms of the preceding series m, $2m$, $3m$, $4m$, + &c.

The 4th order is in like manner derived from the 3d order 1, 4, 10, 20, 35, &c. being the sums of the 3d order 1, 3, 6, 10, &c.; and generally m, $4m$, $10m$, $20m$, $35m$, &c. and so throughout the whole superior orders of numbers.

We may remark, however, that the multiple m remains constant throughout all the orders of numbers, and that it is only the coefficients which undergo any change; and that these coefficients are themselves the terms of the successive orders of 1. So we have only to inquire a little more particularly into the formation of the several orders of an infinite series of units.

Now we shall find that these may be obtained by multiplication as well as by addition.

Thus the series	1, 1, 1, 1, 1, 1, 1, &c.
multiplied by	1, 1, 1, 1, 1, 1, 1, &c.
is	1, 1, 1, 1, 1, 1, 1, &c.
	1, 1, 1, 1, 1, 1, &c.
	1, 1, 1, 1, 1, &c.
	1, 1, 1, 1, &c.
	1, 1, 1, &c.
	1, 1, &c.
	1, &c.
Sums,	1, 2, 3, 4, 5, 6, 7, &c. the natural numbers.

In like manner, the natural numbers, or 2d order of 1, multiplied by the 1st order of 1, produces the 3d order of 1, or the trigonal numbers. Thus,

	1,	2,	3,	4,	5,	6,	7,	&c.
multiplied by	1,	1,	1,	1,	1,	1,	1,	&c.
gives	1,	2,	3,	4,	5,	6,	7,	&c.
		1,	2,	3,	4,	5,	6,	&c.
			1,	2,	3,	4,	5,	&c.
				1,	2,	3,	4,	&c.
					1,	2,	3,	&c.
						1,	2,	&c.
							1,	&c.
* Sums,	1,	3,	6,	10,	15,	21,	28,	&c. the trigonal numbers.

And so forth throughout the whole succeeding orders of numbers.

But if we represent any number by the units of which it is

* The application of this table to find, by the multiplication of its columns, the series of every order of numbers whatsoever, is shewn in a subsequent chapter.

composed, and raise it in the same way to its successive powers by the successive multiplication of the units contained in its root, it is evident that there must be a relation between its powers thus obtained, and the order of numbers of the same degree; the one being the power of an *infinite* series of units, and the other of a *finite* series of units.

We shall immediately perceive this relation, by raising the series of natural numbers, represented by the units contained in them, to the square.

Thus, 2, if represented by 1 + 1, when raised to the square, is

$$\begin{array}{rl} & 1+1 \\ \times \text{ by } & 1+1 \\ \hline = & 1 \quad 1 \\ & \quad 1 \quad 1 \\ \hline & 1, + \bar{2}, + 1 = 4 \end{array}$$

3 raised to the square is $1+1+1$

multiplied by $1+1+1$

$$\begin{array}{rl} & 1, \quad 1, \quad 1 \\ & \quad 1, \quad 1, \quad 1 \\ & \quad\quad 1, \quad 1, \quad 1 \\ \hline \text{Sum,} & 1+2+\bar{3}+2+1 = 9 \end{array}$$

Four units raised to the square, in like manner produce the series, 1, 2, 3, 4, 3, 2, 1 = 16; being as many terms of the series of the 2d order as there are units in the root, plus the same number of terms less one term, the root standing always as the *middle* term of the double series. We obtain, therefore,

this formula for squares, that they are the sum of the series of natural numbers from unity upwards, including the root, plus the same series upwards to the root less one ; or, as the 3d order or trigonal numbers are the sums of the natural numbers, we obtain this rule, that any square, as x^2, is equal to the trigonal of x, plus the trigonal of $(x-1)$, or to the x^{th} term of the 3d order, plus the term $(x-1)$, of the same order, which may be written thus, $x^2 = \binom{\text{T.}\,x}{3} + \binom{\text{T.}\,x-1}{3}$ the T. x or (T. $x-1$), denoting the particular term, and the figure placed below, expressing the order of numbers to which the term belongs.

We observe, farther, that the whole number of terms of the 2d order, composing any square, x^2 are equal to $(2x-1)$; thus, there were $2\times 2-1=3$ terms in the square of 2, viz. 1, 2, 1; $2\times 3-1=5$ terms in the square of 3, viz. 1, 2, 3, 2, 1 ; seven terms in the square of 4, &c.

Theorems for Cubes.

Proceeding in the same way by raising a finite series of units to the cube, I found the whole terms of the 3d order, which were equivalent to the cube of the root x, to be $(3x-2)$ terms, whereof x terms, counting from both ends of the series, or $2x$ terms, were regular terms of the 3d order, and $x-2$ terms defective in the middle of the series ; and which middle terms were rendered defective by the subtraction of a series of the 3d order of 3, or of the 3d order of 1×3.

An example will best explain this: Supposing x to be 6, then $(3x-2)$ terms are $3\times 6 = 18-2 = 16$ terms of the 3d order, which go on to increase to the middle term, and then decrease by the same progression. The regular series of the 3d order would therefore be,

1, 3, 6, 10, 15, 21, 28, 36, 36, 28, 21, 15, 10, 6, 3, 1, but there are $x-2=6$ $-2=4$ middle terms defective, there being this series to be subtracted (1, 3, 3, 1,) $\times 3 =$ 3, 9, 9, 3, so the series becomes,

$=1, 3, 6, 10, 15, 21, 25, 27, 27, 25, 21, 15, 10, 6, 3, 1 = 216$

Whence we have this formula for cubes, in terms of the 4th order of numbers, which are the summations of the terms of the 3d order; thus $x^3 = 2$ term $\left(\frac{3x-2}{2}\right)$ of the 4th order, $-2 \times 3 = -6$ term $\left(\frac{x-2}{2}\right)$ of the same order.

Thus, the $\left(\frac{3x-2}{2}\right)$ term of the 4th order corresponding to $x = 6$, is the 8th term of the 4th order $= 120$, which $\times 2$ is - - 240

And the $\left(\frac{x-2}{2}\right)$ term $=$ 2d term of the same order $= 4 \times 6 = 24$, which being subtracted 24

leaves $6^3 = 216$

After the same manner I found that the whole terms of the 4th order, which were equivalent to the 4th power of a finite series of units x was $(4x-3)$ terms, whereof $2x$ terms were regular terms of the 4th order, and $2x-3$ terms defective, the terms to be deducted being $2x-3$ terms of the same order $\times 4$, (the coefficient of the 2d term of a binomial raised to the 4th power). Thus, supposing $x = 4$, the whole

terms $= 4x-3 = 13$ terms, which divided by 2 gives 7 for the middle term of the series. The series equivalent to 4 is therefore, 7 + 6 terms of the 4th order, viz.

1, 4, 10, 20, 35, 56, $\overline{84}$, 56, 35, 20, 10, 4, 1

diminished by $(2x-3) = 8-3 = 5$ terms of the same order $\times$ 4. — 4, 16, 40, 16, 4

Leaving this series,

1, 4, 10, 20, 31, 40, 44, 40, 31, 20, 10, 4, 1 = 256

Whence we have this formula, in terms of the 5th order of numbers, which are the summations of the 4th order, x^4 = 2 term $\left(\frac{4x-3}{2}\right)$ + term $\left(\frac{4x-3}{2}-1\right)$ of the 5th order — term $\left(\frac{2x-3}{2}\right)$ — term $\left(\frac{2x-3}{2}-1\right)$ of the same order $\times$ by 4.

The number of terms of the 5th order equivalent to a 5th power, were found in the same way to be equal to $(5x-4)$ whereof x terms, beginning at both ends of the series, are regular, $= 2x$ complete terms of the 5th order, and, from the remaining terms being $(2x-4)$ middle terms, there is to be subtracted $(2x-4)$ terms of the 5th order, beginning with unity $\times$ by 5, the coefficient of the 2d term of a binomial raised to the 5th power, which are also regular or complete for x terms, beginning from each end of this 2d series, but the remaining middle terms being $(x-2)$ terms, are again to be increased by $(x-2)$ terms of the same order, $\times$ by 10, the coefficient of the 3d term of a binomial, raised to its 5th power.

We have, therefore, this formula in terms of the 6th order,

being the summation of the 5th order, $x^5 = 2$ term $\left(\frac{5x-4}{2}\right)$ of the 6th order $-$ 2 term $\left(\frac{3x-4}{2}\right)$ of the same order $\times$ 5, $+$ 2 term $\left(\frac{x-4}{2}\right)$ of the same order $\times$ by 10.

Take for example $x = 5$. Here the whole terms are $(5x-4) = (5 \times 5)-4 = 21$ terms, which divided by 2 gives 10 and 11 for the terms of the 6th order $= 2002 + 3003 =$ 5005
$-$ the 5th and 6th terms of the same order $\times$ by $5 =$
$126 + 252 = 378 \times 5 =$ 1890

Remainder, 3115

which is to be increased by 5—4 terms of the same order, $\times$ $10 = 0 + 1 \times 10 =$ 10

Sum, $= 5^5 =$ 3125

The 6th powers were found to consist of $(6x-5)$ terms of the 6th order, whereof x terms from each end $= 2x$ terms were complete, the remaining $4x-5$ terms were diminished by $(4x-5)$ terms of the same order, $\times$ 6, the coefficient of the 2d term of a binomial raised to the 6th power, which are also regular or complete, for x terms from each end or $2x$ terms of this 2d series, leaving still $2x-5$ middle terms to be increased by as many terms of the same order $\times$ 15, the coefficient of the 3d term of a binomial raised to the 6th power.

The 7th powers consisted of $(7x-6)$ terms, whereof $2x$ terms were complete, leaving $(5x-6)$ terms, to be diminished by the same series $\times$ 7, whereof $2x$ terms were again complete—leaving $(3x-6)$ terms to be increased by the same series $\times$ 21, the coefficient of the 3d term of a binomial raised to the 7th power, whereof $2x$ terms were again complete, but still

leaving a remainder in the middle of $(x-6)$ terms, to be diminished by the same series $\times$ 35, the coefficient of the 4th term of a binomial raised to the 7th power.

And, finally, calling the power m, I found that x^m consisted of $(mx-(m-1)$ terms of the mth order,* whereof $2x$ terms were complete, leaving the whole terms $-2x$ to be diminished by the terms of the same order multiplied by m, the coefficient of the 2d term of a binomial raised to the power m, increased by the whole terms $-4x$, of the same order, $\times \frac{m \cdot m-1}{1 \cdot 2}$ the coefficient of the 3d term of a binomial raised to the power m, diminished by the whole terms $-6x$, of the same order $\times \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3}$ the coefficient of the 4th term of a binomial raised to the power m, and so on, diminishing and increasing alternately until the multiple of x, to be deducted from the whole terms of the mth power, was greater than m. The general law may be expressed thus:

$$= \left(\frac{2\mathrm{T}mx-(m-1)}{m+1}\right) - m\left(\frac{2\mathrm{T}mx-(m-1)-2x}{m+1}\right)\frac{+\,m \cdot m-1}{1 \cdot 2}$$

$$\left(\frac{\mathrm{T}\,mx-(m-1)-4x}{m+1}\right) - \text{\&c. or calling } \left(\frac{2\,\mathrm{T}\,mx-(m-1}{m+1}\right)$$

$$= \mathrm{A} \text{ we have } \mathrm{A} - m\,(\mathrm{A}-2x) + \frac{m \cdot m-1}{1 \cdot 2}(\mathrm{A}-4x) -$$

$$\frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3}(\mathrm{A}-6x) + \text{\&c.}$$

until m is exhausted.

* This expression is the same with $mx+1-m$, but I have written it as in the text, to shew the nature of the law, which in words would be "Multiply the index "of the power by the root, and subtract the index of the power less one, for the "number of terms of the order m."

I subjoin an example, where $m=10$, and $x=5$; here the whole terms being $mx-9=10\times5-9=2\times\frac{41}{2}$ terms of the 10th order = 20th + 21st terms of the 10th order, whereof 2×5 being complete, leaves the 15th + 16th terms of the 11th order $\times$ 10, and 2×5 terms being again complete, there is to be added the 10 + 11th terms of the order $\times$ 45. And, again, subtracting 2×5 terms, there remains to be deducted the 5th + 6th terms $\times$ 120, subtracting again 2×5 terms there remains the 0 + the 1st terms to be added $\times$ 210; so the whole calculation is

20+21st terms 11th order = 20030010 + 30045615 =		50075025
−15th + 16th terms × 10 = 1961256 + 3268760 =	5230016 × 10	52300160
	Leaves —	2225135
+10th+11th terms × 45 = 92378 + 184756 =	277134 × 45 =	12471030
	Remains difference, = +	10245895
−5th+6th terms × 120 = 1001+3003 = 4004 × 120 =		480480
	Leaves +	9765415
+ 0 + 1st term × 210 = 1 × 210 =		210
	Sum = 5^{10} =	9765625

The application of these theorems is peculiarly convenient for finding the real or approximate roots of the higher equations, by the mere inspection of a table of the orders of numbers. My design, in this memoir, will not allow me to enter much into detail; but I subjoin a few examples.

1. *Application of the Formula for Squares.*

$$x^2 = \text{Trig. of } x + \text{Trig. } x-1 = T^x_3 + \left(\begin{smallmatrix} Tx-1 \\ 3 \end{smallmatrix}\right)$$

The difference of two trigonal numbers, or numbers of the 3d order, being the natural number or root, and the sum of two trigonals, or numbers of the 3d order, being the square of the root, half the square must be equal to $T\left(\begin{smallmatrix} x-1 \\ 3 \end{smallmatrix}\right) + \frac{1}{2}x$, or

$T\frac{x}{3} - \frac{1}{2}x$.

Therefore, if we divide a square by 2, the nearest trigonal or number of the 3d order above or below it, will equally shew the root, the greater being the trigonal of the root, and the less the trigonal of the root —1.

Thus $\frac{144}{2} = 72$, the next higher trigonal is 78, the trigonal of 12; the root, and the immediate lower trigonal is 66, the trigonal of 11 = the root —1.

Farther, we see that 78 is 6, or one-half of 12 above 72, and 66 is 72—6, the half of 12 below it. And in the case of incommensurable roots, the nearer that half the power is to the higher or lower trigonal, shews the degree of its approximation to the integer root above or below it: Thus 170, divided by 2, is 85, the next higher trigonal is 91, and the immediate lower trigonal is 78; so the root is rather greater than 13; for if it had been exactly 13, it would have $91 - 6\frac{1}{2} = 84\frac{1}{2}$, in place of 85, being $\frac{2}{13}$ or $\frac{1}{26}$ above it, so $13\frac{1}{26}$ should be very nearly the root of 170: $13\frac{1}{26} \times 13\frac{1}{26} = 169 + 1 + \frac{1}{26^2} =$, exceeding 170 by only $\frac{1}{676}$ part.

Required the square root of 2: Annexing two cyphers, we have $\frac{200}{2} = 100$; the nearest higher trigonal number is $105 =$ trig. of 14; but if x had been 14, $T\,\frac{14}{3} - \frac{1}{2}x$ must have been $105 - 7 = 98$, instead of 100; so x is greater than 14 by $\frac{2}{14}$ nearly, trying $x = 14\frac{1}{7}$; $x^2 = 196 + 4 + \frac{1}{49} = 200 + \frac{1}{49}$. If we annex four cyphers, we shall have $\frac{20,000}{2} = 10,000$, the nearest higher trigonal number is 10,011 the trig. of 141; but if x had been 141, $T\,\frac{141}{3} - \frac{1}{2}x$ would have been $10,011 - 70\frac{1}{2} = 9940\frac{1}{2}$ instead of 1000; so x is greater than 141, by nearly $\frac{59\frac{1}{2}}{141} = \frac{3}{7}$ nearly, making $x = 141 + \frac{3}{7}$ we have $x^2 = 20,002 + \frac{9}{49}$.

Required $\sqrt{50}$: Annexing 4 cyphers, we have $\frac{50,000}{2} = 2500$; the nearest higher trigonal is 25,000 being the trigonal of 224, but if x had been 224, $T\,\frac{224}{3} - \frac{1}{2}x$ must have $25,200 - 112 = 25,088$ in place of 25000, so 224 is too great by nearly $\frac{88}{224} = \frac{11}{28}$, making $x = 224 - \frac{11}{28} = 223 + \frac{17}{28}$ we have $x^2 = 5,0000 + \frac{289}{784}$ which gives the power true to 4 places of decimals.

Finally, we derive from the formula $x^2 = T\,\frac{x}{3} + T\,\frac{x}{3} - 1$ this general rule for the involution and evolution of squares:

To involve any number to its square, "Find the trigonal of "the number itself, and also that of the immediate lower num- "ber, the sum of these trigonals will be its square;" and to evolve a square,—"Divide by 2, and look for the next higher tri- "gonal, which is the trigonal of the root, or of the approximate "root, if the number is not the square of an integral number."

I proceed now to the application of the general formula to the evolution of cubes.

Our formula is $x^3 = 2$ term $\left(\frac{3x-2}{2}\right)$ of the 4th order -6 term $\left(\frac{x-2}{2}\right)$ of the same order.

Here the first term of the formula being by much the greater, we need only attend to it in order to find the root. We divide the given cube by 2, and look in the table of the 4th order for the nearest higher tabular number; which we make $= \frac{3x-2}{2}$, and observe the number of the term, and doubling the number of the term, we have twice the term $= 3x-2$, and consequently 2 term $+ 2 = 3x$; twice the term increased by 2, must therefore be divisible by 3, which enables us to find the root.

Ex.—Required the cube root of 1728, here $\frac{1728}{2} = 864$; the next higher number of the 4th order is 969, being the 17th term; doubling the number of the term we have $17 \times 2 = 3x-2 = 34 + 2 = 3x = 36 = 3x$ and $x = 12$, the root.

Required $\sqrt[3]{3}$.

Here we must annex three times as many cyphers as we desire to have decimal places in the root; annexing three cyphers, we have $\frac{3000}{2} = 1500$; the next higher number of the

4th order is 1540, the 20th term ; doubling the number of the term we have $20 \times 2 = 40 = 3x - 2$ and $42 = 3x$, consequently $x = 1.4$ the approximate root ; annexing 6 cyphers we have $\frac{3000,000}{2} = 1500,000$; the next higher number in the table of the 4th order is 1521520 the 209th term; doubling the number of the term we have $209 \times 2 = 418 + 2 = 3x = 420 = 3x =$ and x 1.40.

The evolution of 4th powers is performed, by this method, with the same ease as their involution.

The formula is $x^4 = 2$ term $\left(\frac{4x-3}{2}\right) - 2$ term $\left(\frac{x-3}{2}\right)$ of the 5th order.

Attending only to the first part of this equation, we have this rule, " Divide the power by 2, and look for the next higher " term in the table of the 5th order, doubling the number of " this term, and adding 3, and dividing by 4, we shall obtain x; " or, if the number so found is not divisible by 4, we must take " the nearest number that is a multiple of 4."

Thus, required $\sqrt[4]{81}$ dividing by 2, we have 40, and looking in the table, we find 70 to be the first number of the 5th order, which exceeds 40, and we observe it is the 5th term we have, therefore, this equation, $5 \times 2 + 3 = 4x$, $= 13$, which is not divisible by 4, but 12 is, and shews the root to be 3.

Required $\sqrt[4]{625}$, dividing by 2, we have 312 ; the first number that exceeds it in the table is the 8th term 330 ; so we have $8 \times 2 + 3 = 4x$, $19 = 4x$; but 19 not being divisible by 4, we look for the number nearest to it that is a multiple of 4, which being 20, gives $x = 5$, the root.

We may observe, that, in the *even* powers, we never can have the exact number by taking the double of the number of

the term, because $2x-1$, $4x-3$, $6x-5$, &c. cannot be an even number, and consequently cannot be divisible by 2. We ought therefore to take one more or less than twice the number of the term, which will give us the exact root: Thus, in the 1st example, if, in place of twice the 5th term, we had taken the 4th + the 5th term $\times 3 = 4x$, we should have had $4+5+3 = 12 = 4x$; and in the 2d example, in place of twice the 9th term, if we had taken the 9th and 8th terms, then as $8+9+3 = 20 = 4x$, we should have obtained $x = 5$.

For 5th powers, and other higher powers, we proceed in the same way; only, when the powers are high, it is unnecessary to divide the power by 2, as the tabular numbers of the higher orders differ so widely, that even one-half the power has little effect on the number of the term; and as double the term, increased by the index of the power —1, must always be divisible by the index of the power, the higher the index of the power is, we are brought the more immediately to the true root.

I shall here conclude what I had to observe upon the theorems in terms of the orders of numbers derived from the synthesis of powers. They are chiefly valuable, as shewing the perfect relation between the powers and the orders of numbers,—a relation, however, subject, as I have reason to think, to some farther modifications than those which have been mentioned; for, upon trial with a very small number, I found that, after the 16th power, a new series, apparently of 100,000, came into the calculus, the laws of which I could not investigate without making tables to a considerable extent, which, for such high powers, would have been a work of more labour than I could, at the time, bestow upon it. The reason of this anomaly may perhaps be, that there appears to be a new series of the natural numbers, which come in after every interval of

four powers; and these may again be subject to a new change after 4×4 or 16 powers.

I shall only remark farther, that any other terms in the equation, beside the highest power itself, can little effect the finding of the approximate root, by a method which has reference to such great differences as subsist between the terms of the higher orders of numbers; and consequently, that these methods will apply almost equally to affected as to pure equations of the higher degrees.

CHAPTER II.

ADDITIONAL THEOREMS, DERIVED FROM THE SYNTHESIS OF POWERS.

Additional Theorems respecting Square Numbers.

1. We immediately derive as corollaries from the formula $x^3 = {}^{\mathrm{T}}_{3}x + \binom{\mathrm{T}\,x-1}{3}$ these other formulæ, that $x^3 = {}^{2\,(\mathrm{T}\,x)}_{3} - x$, and $2\binom{\mathrm{T}\,x-1}{3} + x$, x itself being the difference between ${}^{\mathrm{T}}_{3}x$ and $\binom{\mathrm{T}\,x-1}{3}$.

2. We observed, in treating of the orders of numbers, that the series of odd numbers were equal to $2\,(x-1)+1$, or $2\binom{\mathrm{T}\,x-1}{2}+1$, the sum of which series is $2\binom{\mathrm{T}\,x-1}{3} + x$.

But if we sum the odd numbers 1, 3, 5, 7, 9, &c. we shall have the series of squares 1, 4, 9, 16, 25, &c.; consequently, the series of squares are equal to $2\binom{\mathrm{T}\,x-1}{3}$, $+ x$ the formula already obtained as a corollary from $x^3 = {}^{\mathrm{T}}_{3}x + {}^{\mathrm{T}\,x-1}_{3}$.

3. If we sum the series of squares, we obtain the series 1, 5, 14, 30, 55, &c. and by raising the formula $2\,\mathrm{T}\,{}^{x-1}_{3} + x$ to a new order, we have $2\,\mathrm{T}\,{}^{x-1}_{4} + \mathrm{T}\,{}^{x}_{3}$ = the sums of the series of squares.

We shall prove this by an example.

Required the sum of the first ten squares. Here $x - 1 = 9$, and the 9th term of the 4th order being 165, this $\times\ 2 =$ 330, and increased by the trigonal of $x = 55$, gives 385 for the sum, which we shall try,

Series of roots $= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
Series of squares $= 1, 4, 9, 16, 25, 36, 49, 64, 81, 100$
Sums $= 1, 5, 14, 30, 55, 91, 140, 204, 285, 385$

4. The sums of the squares may be found otherwise, the formula $\mathrm{T}\,{}^{x}_{3} + \left(\mathrm{T}\,{}^{x-1}_{3}\right)$ gives $\mathrm{T}\,{}^{x}_{4} + \left(\mathrm{T}\,{}^{x-1}_{4}\right)$: Taking the same example, the 10th term of the 4th order is 220, and the 9th term = 165, their sum is 385. The formula $2\,\mathrm{T}\,{}^{x}_{3} - x =$ x^2, gives us also for the sum of the squares $2\,\mathrm{T}\,{}^{x}_{4} - \mathrm{T}\,{}^{x}_{3}$. Taking still the same example, we have $\mathrm{T}\,{}^{x}_{4} = 220 \times 2 = 440 - \left(\mathrm{T}\,{}^{x}_{3}\right) = 55 = 385$.

The sums of the squares may also be found in terms of the 5th order, for the squares themselves will be found equal to $\mathrm{T}\,{}^{x}_{4} - \left(\mathrm{T}\,{}^{x-2}_{4}\right)$. Thus, 5^2 = the 5th term of the 4th order—the 3d term of the same order = $35 - 10 = 25$, consequently the sums of the squares $= \mathrm{T}\,{}^{x}_{5} - \left(\mathrm{T}\,{}^{x-2}_{5}\right)$.

Thus taking the former example, where $x = 10$, we have $715 - 330 = 385$ for the sum of the first ten squares.

5. If it were desired to find the sum of any number of terms of the series of the odd or of the even squares, these might be obtained by means of the following formulæ.

Formula for the odd squares: $(2x+1)^2 = 8\left(\begin{smallmatrix}\mathrm{T}\\ 3\end{smallmatrix}^{x-1}\right) + 1$, and consequently the sum of this series is $= 8\left(\begin{smallmatrix}\mathrm{T}\\ 4\end{smallmatrix}^{x-1}\right) + x$.

The formula for the even squares being $= 8\left(\begin{smallmatrix}\mathrm{T}\\ 3\end{smallmatrix}^{x-1}\right) + 4x$, their sums must be equal to $8\left(\begin{smallmatrix}\mathrm{T}\\ 4\end{smallmatrix}^{x-1}\right) + 4\,{}^{\mathrm{T}}_{3}x$.

We shall give one or two examples of the application of these formulæ.

Required the sum of the first five odd squares.

Here $x = 5$ and the formula being $8\left(\begin{smallmatrix}\mathrm{T}\\ 4\end{smallmatrix}^{x-1}\right) + x$, we must take the 4th term of the 4th order $= 20 \times 8 + 5 = 165$, which we shall find equal to this series, $1 + 9 + 25 + 49 + 81 = 165$.

Required the sum of the series of odd squares, beginning with the square of 7, and ending with the square of 21.

Here $x = 11$, (for 21 is the 11th term of the series of odd numbers,) and by the formula we take the 10th term of the 4th order, which we are to multiply by 8, and add the term, which is 11: 10th term of the 4th order $= 220 \times 8 =$ 1760

$+ x =$ 11

Sum of the series of 11 squares = 1771

From which we must deduct the 3 first squares, (7 being the 4th odd number,) so $x = 3$, and by the formula we have $x - 1 = (3-1) =$ 2d term of the 4th order

$$\times 8 + x = 4 \times 8 + 3 = \begin{array}{r} 1771 \\ 35 \end{array}$$

Sum of series 7^2, 9^2, 11^2, 13^2, 15^2, 17^2, 19^2, $= 1736$
$= 49+81+121+169+225+289+361+441 = 1736$

Required the sum of the first 7 even squares.

Applying the formula $8\binom{T\, x-1}{4} + 4\,{}^{T}_{3}x$ where $x = 7$ we have the $(7-1) = 6$th term of the 4th order $\times 8 +$ 7th term of the 3d order $\times 4 = 56 \times 8 + 28 \times 4 = 448 + 112 = 560 = \overset{2^2}{4} +$
$\overset{4^2}{16} + \overset{6^2}{36} + \overset{8^2}{64} + \overset{10^2}{100} + \overset{12^2}{144} + \overset{14^2}{196} = 560.$

6. These formulæ for the summation of the alternate terms of series of squares, enable us to find the terms of the 4th and 5th orders of numbers.

Thus, the series of squares being,	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c.
If we take the sums of the alternate terms, beginning with the odd squares, we have,	1, 10, 35, 84, 165, 286, &c.
And the sums of the other terms, or of series of the even squares,	0, 4, 20, 56, 120, 220, 364, &c.
By adding these series we obtain the 4th order of numbers,	1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, &c.

So that supposing we had a table of the 4th order of numbers to a certain extent, we could find it to double the number of terms by these formulæ. Thus,

If our table of the 4th order did not exceed ten terms, and it were required to find the 19th term of that order, here we ap-

ply the formula $8\left({}^{\mathrm{T}}{}_{4}^{x-1}\right)+x$, where $x = 19$, being the 10th term of the series of odd numbers, so we have $\left({}^{\mathrm{T}}{}_{4}^{x-1}\right)$ = 9th term of the 4th order $= 165 \times 8+10 = 1320+ 10=1330$, the 19th term of the 4th order.

But when we apply these formulæ for finding the terms of the 5th order, we must take the sum of both the odd and even squares, as the 5th order includes the sum of all the terms of the 4th order. The terms of the 5th order $= 8\left({}^{\mathrm{T}}{}_{5}^{x-1}\right)$ $+ 8\left({}^{\mathrm{T}}{}_{5}^{x-1}\right) + 4\,{}^{\mathrm{T}}{}_{4}^{x}+{}^{\mathrm{T}}{}_{3}^{x}=16\left({}^{\mathrm{T}}{}_{5}^{x-1}\right) + 4\,{}^{\mathrm{T}}{}_{4}^{x}+{}^{\mathrm{T}}{}_{3}^{x}$.

Thus having the 1st ten terms of the 5th order, let it be required to find the 20th term. Here $10 = x$ and $x-1 = 9$, the 9th term of the 5th order $= 495 \times 16=7920+ 4 \times 220 + 55$ $= 7920 + 880 + 55 = 8855$, the 20th term.

We cannot, however, apply this rule farther, as the 6th order is the sum of all the preceding terms of the 5th order, and this formula only gives the individual term, and not the series of terms of the 5th order.

7. But any term of a higher order of numbers may be always found from the term of the preceding order, by this theorem, ${}^{\mathrm{T}}{}_{m+1}^{x} = \frac{x}{m}\left({}^{\mathrm{T}}{}_{m}^{x+1}\right)$.

Thus to find the term x of the 4th order from the 3d order, we have ${}^{\mathrm{T}}{}_{4}^{x} = {}^{x}\left(\frac{\mathrm{T}\,x+1}{3}\right)$.

and the rule therefore is, to multiply the trigonal of the root + 1 by one-third of the root.

In the same way, the 6th term of the 4th order is equal to the 7th term of the 3d order $\times$ 6, divided by $3 = 28 \times 2 = 56$;

The 10th term of the 4th order = 11th term of the 3d order $\times \frac{10}{3}$

$= 66 \times \frac{10}{3} = \frac{660}{3} = 220$, &c.

The 5th order may also be found in terms of the 4th order, $\mathrm{T}x_5 = \frac{x}{4}\binom{\mathrm{T}x+1}{4}$.

And the 6th order, in terms of the 5th order, $\mathrm{T}x_6 = \frac{x}{5}\binom{\mathrm{T}x+1}{5}$,

&c. &c. &c.

8. I shall conclude these observations respecting square numbers, with the following theorem, which, although an identical proposition, is yet unexpected when first presented to our consideration in words.

A formula for finding any term of a series of numbers, which, when multiplied or divided by the same number, shall be a square number.

But the formula makes the proposition perfectly evident, $\frac{x^2}{m} \times m$, or divided by m is a square. For multiplying by m, we have x^2, and dividing by m, we have $\frac{x^2}{m^2} = \left(\frac{x}{m}\right)^2$.

To apply this formula; if $m = 2$, we have $\frac{x^2}{2} = \frac{2}{2}\binom{\mathrm{T}x}{3}1 + \frac{x}{2} = \binom{\mathrm{T}x-1}{3} + \frac{x}{2}$, by which we obtain this series,

Series $\binom{\mathrm{T}x-1}{3}$ = 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, &c.

Series $\frac{x}{2}$ = $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, &c.

Series $\binom{\mathrm{T}x-1}{3} + \frac{x}{2}$ = $\frac{1}{2}$, 2, $4\frac{1}{2}$, 8, $12\frac{1}{2}$, 18, $24\frac{1}{2}$, 32, $40\frac{1}{2}$, 50, &c.

The doubles and halves of which are all square numbers.

Thus multiplying by 2 we obtain 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 &c.
or dividing by 2, we have $\frac{1}{2}$, 1, $2\frac{1}{4}$, 4, $6\frac{1}{4}$, 9, $12\frac{1}{4}$, 16, $20\frac{1}{4}$, 25

Where the squares are integral in this series, the truth is sufficiently apparent; and the fractional terms will be found, upon trial, to be the squares of $\frac{1}{2}$, of $1\frac{1}{2}$, of $2\frac{1}{2}$, of $3\frac{1}{2}$, of $4\frac{1}{2}$, &c.

If $\frac{x^2}{m} = \frac{x^2}{3}$ we shall have this formula,

$$\frac{2\binom{Tx-1}{3}+x}{3} = \frac{x^2}{3}$$

$2\binom{Tx-1}{3}$ =	0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, &c.
$+x$ =	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
Sums x^2 = Series x^2 =	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121
Dividing by 3 = $\frac{x^2}{3}$ =	$\frac{1}{3}$, $1\frac{1}{3}$, 3, $5\frac{1}{3}$, $8\frac{1}{3}$, 12, $16\frac{1}{3}$, $21\frac{1}{3}$, 27, $33\frac{1}{3}$, $40\frac{1}{3}$
Dividing again by 3 = $\left(\frac{x}{3}\right)^2$ =	$\frac{1}{9}$, $\frac{4}{9}$, 1, $1\frac{1}{9}$, $2\frac{1}{9}$, 4, $5\frac{4}{9}$, $7\frac{1}{9}$, 9, $11\frac{1}{9}$, $13\frac{4}{9}$

If $m = 4 = \frac{x^2}{4} =$

$\frac{2\binom{Tx-1}{3}+x}{4} - \frac{1}{4}\binom{Tx-1}{3}$	= 0, $\frac{1}{2}$, $1\frac{1}{2}$, 3, 5, $7\frac{1}{2}$, $10\frac{1}{2}$, 14, 18, $22\frac{1}{2}$, $27\frac{1}{2}$, &c.
$+\frac{1}{4}x$	= $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, $1\frac{2}{4}$, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{2}{4}$, $2\frac{3}{4}$,
Series $\frac{Tx-1}{2}+x$ = the series of squares $\left(\frac{x}{2}\right)^2$	$\frac{1}{4}$, 1, $2\frac{1}{4}$, 4, $6\frac{1}{4}$, 9, $12\frac{1}{4}$, 16, $20\frac{1}{4}$, 25, $30\frac{1}{4}$,

Dividing again by 4, we have $\frac{x^2}{16}$ another series of squares = series $\left(\frac{x}{4}\right)^2$; and dividing again by 4, we have a third series of squares $= \left(\frac{x}{8}\right)^2$, and so continually, all squares multiplied or divided by 4 being necessarily squares;

viz. dividing by 4, 4^2, 4^3, &c. we obtain the series of squares of $x, \frac{x}{2}, \frac{x}{4}, \frac{x}{8}, \frac{x}{16}$, &c. respectively.

Or multiplying by 4, 4^2, &c. we obtain the series of squares of $\left\{ x^2,\ (2x)^2\ \ (4x)^2,\ (8x)^2\ \ (16x)^2 \right.$ &c. respectively.

And the same takes place when m is any other square number; for $\frac{x^2}{m^2} \times m^2 = x^2 \times m^2 = m^2 x^2 \times m^2 = m^4 x^2$, which are evidently the squares of x, mx, $m^2 x$, $m^3 x$, &c.—or, dividing by m^2, we have $\frac{x^2}{m^2}$ $\frac{x^2}{m^4}$ $\frac{x^2}{m^6}$ $\frac{x^2}{m^8}$ &c. which it is also obvious are the squares of $\frac{x}{m}$ $\frac{x}{m^2}$ $\frac{x}{m^3}$, &c.

These formulæ afford us the means of finding the summation of a series of fractional numbers.

Thus, where $m=2$, the sum of the series $\binom{\mathrm{T}x-1}{3} + \frac{x}{2}$ is evidently $\binom{\mathrm{T}x-1}{4} + \frac{1}{2}\,{}^{\mathrm{T}x}_{3}$.

Required the first 4 terms of the series $\frac{1}{2}$, 2, $4\frac{1}{2}$, 8, &c. $=10+\frac{10}{2}=15$.

And dividing 15 by 2, we have $7\frac{1}{2}$ for the sum of the series of $\binom{x^2}{2}=\frac{1}{4},\ 1,\ \frac{1}{4}2,\ 4=7\frac{1}{2}$.

And so of any other fractions, where a series of x^2 is the numerator, and m is the constant denominator.

I shall have occasion to treat of this afterwards, in a separate chapter, on the summation of series. In the meantime I proceed to the theorems for cubes.

Theorems for Cubic Equations.

9. It has been already observed, that the series of cubes may be found in terms of the 4th order of numbers, multiplied by 6,

adding the root; the general formula is $6\,\mathrm{T}\binom{x-1}{4} + x$. which affords this very easy rule for the involution of cubes.

"Multiply the term of the fourth order, corresponding to the "number, which is one less than the root, by 6, and add the "root; the sum so found will be the cube of the given num- "ber."

Examples.

Required to raise 17 to the cube.

We take the $(x-1)$ term = the 16th term of the fourth order = 816, and multiplying by 6, we obtain 4896, and adding the given number 17, we have the cube of 17 = 4913, which we shall find correct; for dividing 4913 by 17, we obtain 289 and $\frac{289}{17} = 17$.

If it were required to raise 1.7 to the cube we should proceed in the same manner, attending only to allot three decimal places in the cube for each term in the root; thus, the 16th term is, as before, 816, and multiplying by 6, we have 4896, and allowing three decimal places, 4.896, to which we add the root with three decimal places, 0.017, so we have for the cube, as above, . . . 4.913, and the proof will be the same as before. *

* It may be remarked in general here, that when the root, or any of the lower powers, fall to be included as constituent parts of a higher power, as in the preceding formula, where x^3 is made equal to $6\,\mathrm{T}\binom{x-1}{4} + x$, these inferior powers, such as the term x in this formula, are not to be considered as single or independent quantities, but as having special relation to the higher power itself; the root x, therefore, which is to be added in the above example to make up the cube of which it is a

10. The same formula $x^3 = 6\,\mathrm{T}\left(\frac{x-1}{4}\right)$ affords a rule for the evolution of cubes, equally convenient as that which has been derived from it, for their involution; viz. "Divide the given "cube by 6, and look in the table of the fourth order for the "next higher number, the corresponding term, or natural num- "ber, is the root sought."

Examples.

Required $\sqrt[3]{1728}$; dividing 1728 by 6, we obtain 288; the next higher number of the 4th order is 364, and the corresponding term is 12, the root required.

Required $\sqrt[3]{217081801}$; dividing by 6, we obtain $36180300\frac{1}{6}$. Looking in the table we find the next higher number to be 36361101 being the 601st term; so 601 is the root required.

Required $\sqrt[3]{2}$; dividing 2 by 6, we obtain .3333333333, &c. and we allow three times as many places for decimals as we intend to have decimal places in the root. Thus, if we desire 3 decimal places in the root, we must have 9 places in the power. So we look in the table for the next higher number of 9

part, is to consist of the same number of decimal places as the cube itself, or of three decimal places, which number of decimals belongs to the cube, and not of one decimal place only, which would belong to the root x, if single and unconnected with the cube or third power x^3.

It would be different, if, besides the simple power belonging to the perfect cube itself, there were other terms of x to be superadded; thus, if it were required to find the sum of $x^3 + 6x$, making $x = 1.7$, we should find x^3, as before, $=$ 16th term of the fourth order $\times$ 6 $+ x = 4.896 + .017$, allowing three decimal places; but when we added the other term, $6x$, this being independent of the cube itself, we should add to the cube $1.7 \times 6 = 10.2$ and not $.017 \times 6 = .102$.

figures of the 4th order, which we shall find to be .333.395.790, and which is the 1260th term; 1,260 is therefore the approximate cube root of 2.

Another Theorem for Cubes.

11. Since squares are the sums of the odd numbers, we might represent a square in terms of the odd numbers, thus:

$2^2 = 1 + 3$

and multiplying by $1 + 1$ we obtain

$$\begin{array}{r} 1+1 \\ \hline 1,\ 3 \\ 1,\ 3 \\ \hline 2^3 = 1,\ 4,\ 3 = 8 \end{array}$$

which may be written thus, beginning with the middle term, $2^2 = 4 + (1 + 3) = 4 + 4 = 2^2 \times 2$.

The square of 3 being 1, 3, 5, if we multiply by $1 + 1 + 1 = 3$, we obtain

$$\begin{array}{r} 1,\ 3,\ 5 \\ 1,\ 3,\ 5 \\ \hline 1,\ 4,\ 9,\ 8,\ 5, \end{array}$$

which we may write, beginning with the middle term, thus, $9, + (1 + 8), + (4 + 5), = 9 + 9 + 9 = 3^3 \times 3$.

This is a curious demonstration of the identity of the results of addition and multiplication, and also of the relation of the cubes and the series of square numbers; but it does not appear to be attended with any other advantage.

Theorems for the Summation of Cubes.

12. Since any cube is equal to $6\left(\frac{Tx-1}{4}\right)+x$, the formula for the sums of the cubes will be $6\left(\frac{Tx-1}{5}\right)+\frac{Tx}{3}$.

Required the sum of the three first cubes. Here $x=3$, and

$x-1=2$, and 6 times the 2d term of the 5th order is $5\times 6=30$, $+{}^{\mathrm{T}x}_{3}=6$; so the answer is $30+6=36=1+8+27=36$.

Required the sum of the 7 first cubes. Here $x-1=6$; the 6th term of the 5th order is $126\times 6=756+{}^{7}_{3}=756+28=784=1+8+27+64+125+216+343=784$.

13. But as the sums of the cubes are equal to the squares of the trigonal numbers, this formula must also apply for finding the squares of the trigonal numbers; thus, in the above examples, x was assumed equal to 3 and to 7, and the answers 36 and $784=6^2$ and 28^2 the squares of the 3d and 7th trigonal numbers.

It is a farther consequence of this law, that $6\left({}^{\mathrm{T}x-1}_{6}\right)+{}^{\mathrm{T}x}_{4}$ must be the theorem for finding the sums of the squares of the series of trigonal numbers, or for the sums of the sums of the series of cubes.

Required the sum of the squares of the first four trigonal numbers. Here $x-1=3$, and the third term of the 6th order being 21, we have $21\times 6+20=146=1^2+3^2+6^2+10^2=1+9+36+100=146=$the series of four cubes summed twice over, thus

1, 8, 27, 64 = series of cubes =
1, 9, 36, 100 = their sums = squares of trig. nos.
1, 10, 46, 146 = their sums = sums of squares of trig. numbers.

Theorems for 4th Powers.

14. I found that the series of 4th powers might be obtained by the multiplication of the following series of factors, by 12, as their common factor, adding the square of the root,

Series of factors $\overset{1}{0}, \overset{2}{1}, \overset{3}{6}, \overset{4}{20}, \overset{5}{50}, \overset{6}{105}, \overset{7}{196}, \overset{8}{336}, \overset{9}{540}, \overset{10}{825}$, &c.

Thus, $1\times12+4=16=2^4$

$6\times12+9=81=3^4$

$20\times12+16=256=4^4$ &c.

This series of factors may be found in several ways:

1st, From the series of odd numbers, thus

Series of odd numbers $= 1, 3, 5, 7, 9, 11$, &c.

Sums = series of squares $= 2\,{}^{\mathrm{T}}_{3}{}^{x-1} + x \Big\} = 1, 4, 9, 16, 25, 36$, &c.

Sums $= 2\,{}^{\mathrm{T}}_{4}{}^{x-1} + {}^{\mathrm{T}}_{3}{}^{x} \Big\} = 1, 5, 14, 30, 55, 91$, &c.
sums of the squares,

Sums $= {}^{\mathrm{T}}_{5}{}^{x-1} + {}^{\mathrm{T}}_{5}{}^{x} \Big\} = 1, 6, 30, 50, 105, 196$, &c.
series of factors.

Thus if $x=1$, $x^{\overline{5}}1 = 0$, and the formula gives 1

$x=2$, $x^{\overline{5}}1 = 1\times2+4 =$ 6

$x=3$, $x^{\overline{5}}1 = 5\times2+10 =$ 20

$x=4$, $x^{\overline{5}}1 = 15\times2+20 =$ 50 &c.

2. The same series of factors may be also found from any of the other formulæ for the summation of squares; thus ${}^{\mathrm{T}}_{4}{}^{x} + {}^{\mathrm{T}}_{4}{}^{x-1}$, gives ${}^{\mathrm{T}}_{5}{}^{x} + {}^{\mathrm{T}}_{5}{}^{x-1}$, where,

If $x=1$ the formula gives $0+1=1$

$=2$ . . $5+1=6$

$=3$ . . $15+5=20$

$=4$ . . $35+15=50$ &c.

Or the formula $2\,{}^{\mathrm{T}}_{4}{}^{x} - {}^{\mathrm{T}}_{3}{}^{x}$ gives $2\,{}^{\mathrm{T}}_{5}{}^{x} - {}^{\mathrm{T}}_{4}{}^{x}$.

If $x=1$, this formula gives $2-1=1$
$=2$ . . $10-4=6$
$=3$ . . $30-10=20$
$=4$ . . $70-20=50$ &c.

Finally, as the sums of the squares are also equal to $T\frac{x}{5}-\binom{Tx-2}{5}$, we have $\frac{Tx}{6}-T\binom{x-2}{6}$. Thus,

If $x=1$, this formula gives $1-0=1$
$=2$, . . $6-0=6$
$=3$, . . $21-1=20$
$=4$, . . $56-6=50$, &c. *

We have therefore the four following formulæ for 4th powers, attending that x in the formula for the series of factors $=x-1$ in the formula of the power.

1. From $2\binom{Tx-1}{5}+\frac{Tx}{4}$ multiplying by 12, and adding x^2, we obtain $x^4=24\binom{Tx-2}{5}+12\binom{Tx-1}{4}+x^2$

2. From $\frac{Tx}{5}+\binom{Tx-1}{5}$ $x^4=12\binom{Tx-1}{5}+12\binom{Tx-2}{5}+x^2$

3. From $2\frac{Tx}{5}-\frac{Tx}{4}=$ $x^4=24\binom{Tx-1}{5}-12\binom{Tx-1}{4}+x^2$

4. From $\frac{Tx}{6}-\binom{Tx-2}{6}$ $x^4=12\binom{Tx-1}{6}-12\binom{Tx-3}{6}+x^2$

any of which may be very conveniently applied to the involution or evolution of 4th powers; but the 3d formula is perhaps

* This series may likewise be obtained by the multiplication of the trigonal of x by the trigonal of $x-1$ divided by 3.

Thus $\frac{1\times 3}{3}=1$, $\frac{3\times 6}{3}=6$, $\frac{6\times 10}{3}=20$, $\frac{10\times 15}{3}=50$, &c.

the best, as the two terms of which it consists are of different orders, and the coefficient of the higher order is double of that of the inferior order. So that the first term leads at once to the approximate root, if we compare it with the power divided by 24.

15. These theorems immediately furnish us with formulæ for the summation of a series of 4th powers. Thus the 2d formula $12\binom{Tx-1}{5}+12\binom{Tx-2}{5}+x^2=12\binom{Tx-1}{5}+12\binom{Tx-2}{5}+2\binom{Tx-1}{3}+x$, gives for the summation of 4th powers, the theorem $12\binom{Tx-1}{6}+12\binom{Tx-2}{6}+2\binom{Tx-1}{4}+\overset{Tx}{3}$.

Thus, let it be required to find the sum of the four first 4th powers, here $x=4$, and the formula gives,

$$\begin{aligned} 12\binom{Tx-1}{6} &= 12\times 21 = 252 \\ 12\binom{Tx-2}{6} &= 12\times 6 = 72 \\ 2\binom{Tx-1}{3} &= 2\times 10 = 20 \\ \overset{Tx}{3} &= 1\times 10 = 10 \\ \text{Sum} & \qquad\qquad 354 \end{aligned}$$

$= 1 + 16 + 81 + 256 = 354$.

Finally, cubes being, as I have found, the sum of as many terms of the odd numbers as there are units in their roots, beginning with the odd number, denoted by $2\binom{Tx-1}{3}+1$; thus if $x=1$, this formula gives the odd number $= 1$

$x=2$, we have $2+1=3$
$x=3$, we have $6+1=7$
$x=4$, we have $12+1=13$
} for the 1st term of the odd numbers composing the cube.

$1^3 = 1 = 1$ = 1 term.
$2^3 = 8 = 3 + 5$ = 2 terms.
$3^3 = 27 = 7 + 9 + 11$ = 3 terms.
$4^3 = 64 = 13 + 15 + 17 + 19$ = 4 terms.
$5^3 = 125 = 21 + 23 + 25 + 27 + 29$, = 5 terms, &c.

We may multiply these expressions for the cube, by the units in the root, for their 4th power. Thus,

$2^3 = 3 + 5$
$\times$ by $2 = 1 + 1$

$3 , 5$
$3 , 5,$

$3 , 8 , 5,$ which may be written thus:
$8 + (3 + 5) = 8 + 8 = 2^4 = 2^3 \times 2.$

If $x = 3$ then $x^3 = 7 + 9 + 11$
$\times \quad 1 + 1 + 1$

$7 , 9 , 11,$
$7 , 9 , 11,$
$7 , 9 , 11,$

$x^4 = 7 + 16 + 27 + 20 + 11$ which may be written thus, $27 + (7 + 20) + (16 + 11) = 27 + 27 + 27 = 3^3 \times 3.$

This is given only as another instance demonstrative of the identity of multiplication and addition, and of the relation of 4th powers to cubes, which, though we know to be true, yet we cannot easily shew arithmetically otherwise. Here the cube of the root is always the middle term, and the sums of the other terms, taken alternately, are equal to the middle term.

Theorems for 5th Powers.

The series of 5th powers are composed of the following series of factors × 30, adding the root, viz.

Terms,	1,	2,	3,	4,	5,	6,	7,	8,	9;	10.
Series,	0,	1,	8,	34,	104,	259,	560,	1092,	1968,	3333, &c.

This series will be found in terms of the 8th order of numbers by this formula $= {}^{\mathrm{T}}_{8}x - 2\binom{\mathrm{T}\,x-2}{8} + \binom{\mathrm{T}\,-x4}{8}$.

This gives, if $x = 1$ $\quad 1-0 \quad +0 = 1$

$x = 2$ $\quad 8-0 \quad +0 = 8$

$x = 3$ $\quad 36-2\times1 \quad +0 = 34$

$x = 4$ $\quad 120-2\times8 \quad +0 = 104$

$x = 5$ $\quad 330-2\times36 \quad +1 = 259$

$x = 6$ $\quad 792-2\times120+8 = 560$, &c.

which formula, therefore, × 30, adding x, will answer for the involution of 5th powers, remembering that x in the formula of the factors is to be made $x-1$ in the formula of the powers.

Let $x = 6$.

$$30\binom{\mathrm{T}\,x-1}{8} = 30 \times 330 = 9900$$

$$+\ 30\binom{\mathrm{T}\,x-5}{8} = 30 \times 1 = 30$$

$$+\ x = 6$$

$$\text{Sum} = 9936$$

$$-60\binom{\mathrm{T}\,x-3}{8} = -60\times36 = 2160$$

Leaves $6^5 = 7776$, which is correct.

The summation of the series of 5th powers is,

$$30\binom{\mathrm{T}\,x-1}{9} - 60\binom{\mathrm{T}\,x-3}{9} + 30\binom{\mathrm{T}\,x-5}{9} + {}^{\mathrm{T}}_{3}x.$$

Required the sum of the first six, 5th powers; here $x=6$, and

$30\binom{\mathrm{T}\,x-1}{9} = 495 \times 30 = 14850$

$30\binom{\mathrm{T}\,x-5}{9} = 1 \times 30 = 30$

$\mathrm{T}\frac{x}{3} =$. 21

Sum = 14901

$-60\binom{\mathrm{T}\,x-3}{9} = 60 \times 45 = 2700$

Leaves the answer, 12201 $= 1^5+2^5+3^5+4^5+5^5+6^5$

$=1+32+243+1024+3125+7776 =$ - - 12201

6th Powers.

The following series of factors $\times$ 60, and increased by the square of the root, form the series of 6th powers, viz.

0, 1, 12, 68, 260, 777, 1960, 4368, 8856, 16665, &c.

which may be found by the following formula.

$$6\binom{\mathrm{T}\,x-1}{8} - 6\binom{\mathrm{T}\,x-3}{8} + \mathrm{T}\frac{x}{6} - \binom{\mathrm{T}\,x-2}{6}$$

Thus if $x = 1$ we have $0-0+1-0=1$

$x = 2$ $6-0+6-0=12$

$x = 3$ $48-0+21-1=68$

$x = 4$ $216-6+56-6=260$

$x = 5$ $720-48+126-21=777$

$x = 6$ $1980-216+252-56=1960$, &c.

The formula for 6th powers, accordingly, is found by multiplying by 60, and adding the square of the root, attending that x

in the formula for the factors $=x-1$ in the formula for the powers; $x^6 = 360\binom{Tx-2}{8} - 360\binom{Tx-4}{8} + 60\binom{Tx-1}{6} - 60\binom{Tx-3}{6} + 2\binom{Tx-1}{3} + x$, and the sum of a series of 6th powers will be obtained by this formula, $360\binom{Tx-2}{9} - 360\binom{Tx-4}{9} + 60\binom{Tx-1}{7} - 60\binom{Tx-3}{7} + 2\binom{Tx-1}{4} + {}^{T}x_{3}$.

Theorems for 7th Powers.

The series of 7th powers are equal to the following series of factors $\times$ 120, increased by the cube of the root, viz.

Terms, 1 2 3 4 5 6 7 8 9 10

Series, 0, 1, 18, 136, 650, 2331, 6860, 17472, 39852, 83395, &c.

which may be found by the following formula:

$$42\binom{Tx-2}{8} + 14\binom{Tx-1}{6} + {}^{T}x_{4}.$$

Thus if $x = 1$ we have $0 + 0 + 1 = 1$

$x = 2$ $\quad 0 + 14 + 4 = 18$

$x = 3$ $\quad 42 + 84 + 10 = 136$

$x = 4$ $\quad 336 + 294 + 20 = 650$, &c.

so $x^7 = 42 \times 120\binom{Tx-3}{8} + (14 \times 120)\binom{Tx-2}{6} + 120\binom{Tx-1}{4} + 6\binom{Tx-1}{4} + x$.

And the series of 7th powers $=$

$$120\left(+42\binom{Tx-3}{9} + 14\binom{Tx-2}{7} + \binom{Tx-1}{5}\right) + 6\binom{Tx-1}{5} + {}^{T}x_{3}.$$

Theorem for 8th Powers.

The series of 8th powers may be found by means of the following series of factors, $\times 240$, and adding the 4th power of the root, viz.

Terms,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
Series,	0,	1,	27,	272,	1625,	6993,	24010,	69888,	179334,	416625,

&c. and which series of factors are obtained by this formula:

$$168\binom{\mathrm{T}\,x-3}{9} + 84\binom{\mathrm{T}\,x-2}{8} + 42\binom{\mathrm{T}\,x-2}{7} + 21\binom{\mathrm{T}\,x-1}{6} + 2\binom{\mathrm{T}\,x-1}{5} + \binom{\mathrm{T}\,x}{4}.$$

Thus, if

$x=1$ we have	0 +	0 +	0 +	0 +	0 +	1 =	1	
$=2$	0 +	0 +	0 +	21 +	2 +	4 =	27	
$=3$	0 +	84 +	42 +	126 +	10 +	10 =	272	
$=4$	168 +	672 +	294 +	441 +	30 +	20 =	1625,	&c.

And, consequently, the formula for 8th powers will be obtained by multiplying the formula for the factors by 240, and adding the 4th power of the root.

$$240\left(168\binom{\mathrm{T}x-4}{9}+48\binom{\mathrm{T}x-3}{8}+42\binom{\mathrm{T}x-3}{7}+21\binom{\mathrm{T}x-2}{6}+2\binom{\mathrm{T}x-2}{5}+\binom{\mathrm{T}x-1}{4}\right)+x^4 = +24\binom{\mathrm{T}x-2}{5}+24\binom{\mathrm{T}x-1}{4}+2\binom{\mathrm{T}x-1}{3}+x.$$

And for the summation of the series of 8th powers, we have only to increase the orders of the terms 1 degree.

Theorems for 9th Powers.

The series of 9th powers may be found by the following series of factors, $\times$ 240, adding the 5th powers of the root.

Terms, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Series, 0, 2, 81, 1088, 8125, 41958, 168070, 559104, 1614006, 4166250,

which will be found by this formula.

$$1512\binom{\mathrm{T}\,x-3}{10}+630\binom{\mathrm{T}\,x-2}{8}+73\binom{\mathrm{T}\,x-1}{6}+2\,\mathrm{T}\,x_{4}$$

Thus if $x=1$ we have $0+0+0+2=2$
$=2$ $0+0+73+8=81$
$=3$ $0+630+438+20=1088$
$=4$ $1512+5040+1533+40=8125$, &c.

And multiplying by 240, and adding the formula for 5th powers, we obtain the following formula for 9th powers.

$$x^9=240\left(1512\binom{\mathrm{T}x-4}{10}+630\binom{\mathrm{T}x-3}{8}+73\binom{\mathrm{T}x-2}{6}+2\binom{\mathrm{T}x-1}{4}\right)+x^5=+30\binom{\mathrm{T}x-1}{8}-60\binom{\mathrm{T}x-3}{8}+30\binom{\mathrm{T}x-5}{8}+x.$$

And the summation of a series of 9th powers will be had in the same terms, increasing the orders of the terms one degree.

Theorems for 10th Powers.

The series of 10th powers will be obtained by multiplying the following series of factors by 120, and adding the 6th powers, viz. 0, 8, 486, 8704, 81250, 503496, 2352980, 8945664, 29052108, 83325000, &c. which may be obtained by this formula:

$$30240\binom{\mathrm{T}x-4}{11}+15120\binom{\mathrm{T}x-3}{10}+10080\binom{\mathrm{T}x-3}{9}+5040\binom{\mathrm{T}x-2}{8}+876\binom{\mathrm{T}x-2}{7}+438\binom{\mathrm{T}x-1}{6}+16\binom{\mathrm{T}x-1}{5}+8\,\mathrm{T}x_{4}$$

Thus, if $x=1$, we have $0+0+0+0+0+0+0+0+8=8$
$x=2$, $0+0+0+0+0+0+438+16+32=486$
$x=3$, $0+0+0+0+0+5040+2628+80+80=8704$
$x=4$, $0+15120+10080+40320+6132+9198+240+160=81250$
&c.

And the terms of this formula $\times$ 120, and increased by the formula for 6th powers, will give the 10th powers; and increasing the orders of the terms 1 degree, we obtain the formula for the summation of the series of 10th powers.

Beyond the 6th power these formulæ for finding the series of factors have been obtained analytically, though the series of factors themselves were derived from the synthesis of the powers.

CHAPTER III.

THEOREMS DERIVED FROM THE ANALYSIS OF POWERS.

I HAVE observed, in the conclusion of the preceding chapter, that it was only in consequence of the analysis, that I was enabled to find the general formula for the series of factors of the several powers exceeding the 5th degree, though the factors themselves were derived from the synthesis; but I thought it unnecessary to separate the series of factors from the theorems for obtaining them, though derived from another source.

This will, however, enable me to be very short in the statement of the theorems derived from the analysis, as they are in general very similar to those already treated of in the preceding chapter.

Theorems.

I have nothing new to observe upon the squares.

Theorems for Cubes.

We have already given the theorems for involving or evolving cubes, in terms of the 4th order of numbers, adding the root. But we may also find a theorem for them, subtracting the squares.

Thus if from the series of cubes	1, 8, 27, 64, 125, 216, &c.
We subtract the series of squares	1, 4, 9, 16, 25, 36
There remains this series	0, 4, 18, 48, 100, 180, &c.
Taking the 1st difference	4, 14, 30, 52, 80, &c.
2d difference	10, 16, 22, 28, &c.
3d difference	6, 6, 6, &c.

From which we obtain of $x^3-x^2=$
1st a series of 6, 6, 6, 6, 6, &c.
$+$ a series of 4,

Sums	4, 10, 16, 22, 28, 34
Sums again	4, 14, 30, 52, 80, 114
Sums again	4, 18, 48, 100, 180, 294, &c. whence this formula, $x^3-x^2=6\binom{Tx-2}{4}+4\binom{Tx-1}{3}$.

Which we might also have obtained at once; thus,

$$x^3=6\binom{Tx-1}{4}+x=6\binom{Tx-2}{4}+6\binom{Tx-1}{3}+x.$$

Subtracting $x^2 \quad = \quad -2\binom{Tx-1}{3}+x.$

Remains $x^3-x^2 \quad =6\binom{Tx-2}{4}+4\binom{Tx-1}{3}+0$, as above.

There is no form of an equation which we may not manage in the same way: Thus,

$$x^3-3x^2=6\binom{Tx-2}{4}-2x.$$

Example.—Let $x=6$, then x^3	$=$	216
and $-3x^2=$	$-$	108
	Remains	$108=$

$$6\binom{Tx-2}{4}-2x=6\times 20-2\times 6=120-12=108.$$

Or if we add the square of x to the cube of x, we shall have $x^3 = \times 6\binom{Tx-1}{4} + x$,

and $+x^2 = 2\,{}^{Tx}_{3} - x$.

$$x^3 + x^2 = 6\binom{Tx-1}{4} + 2\,{}^{Tx}_{3} + *.$$

If $x=4$, then $x^3+x^2=64+16=6\times10+2\times10=80$.

Given the equation $3x^3+10x^2+6x=189$.

Then $189=18\binom{Tx-1}{4}+3x+20\,{}^{Tx}_{3}-10x+6x=18\binom{Tx-1}{4}$ $+20\,{}^{Tx}_{3}-x$.

Thus if $x=3$ we have $3\times27+10\times9+6\times3=81+90$ $+18=189=18\,{}^{Tx-1}_{4}+20\,{}^{Tx}_{3}-x=18\times4+20\times6-3=72+$ $120-3=192-3=189$.

This method will be particularly convenient where there are several values to be derived from the same equation.

Theorems for 4th Powers.

Taking the successive difference of a series of 4th powers, we arrive at the 3d formula for 4th powers, which has been already given in the synthesis, viz.

$$x^4=24\binom{Tx-2}{5}+12\binom{Tx-1}{4}+x^2.$$

Therefore $x^4-x^2=24\binom{Tx-1}{5}+12\,{}^{Tx-1}_{4}$.

and subtracting $x^3 = -6\binom{Tx-1}{4}+x$, we obtain

$x^4-x^3-x^2 = 24\binom{\mathrm{T}x-2}{5}+6\binom{\mathrm{T}x-1}{4}-x$, and subtracting x

$x^4-x^3-x^2-x=24\binom{\mathrm{T}x-2}{5}+6\binom{\mathrm{T}x-1}{4}-2x.$

Thus, if $x=4$,
$256-64-16-4=172=24\times5+6\times10-8=120+60-8=180-8=172.$

We may adapt our formula to any other form of the equation in the same way; thus,

$$\begin{array}{rrrl} x^4+x^3+x^2+x=x^4 = 24\binom{\mathrm{T}x-2}{5} & +12\binom{\mathrm{T}x-1}{4} & +2\,\mathrm{T}\frac{x}{3} & -x \\ +x^3 & +6\binom{\mathrm{T}x-1}{4} & & +x \\ +x^2 & & +2\,\mathrm{T}\frac{x}{3} & -x \\ +x & & & +x \\ \hline x^4+x^3+x^2+x & 24\binom{\mathrm{T}x-2}{5} & +18\binom{\mathrm{T}x-1}{4} & +4\,\mathrm{T}\frac{x}{3}+* \end{array}$$

Thus, if $x=3$. $81+27+9+3=120=24+18\times4+4\times6=24+72+24=120$.

Theorems for 5th Powers.

By taking the successive differences of a series of 5th powers, we arrive at this formula of their formation, which is different from that given in the synthesis, viz. $x^5=120\binom{\mathrm{T}x-2}{6}+30\binom{\mathrm{T}x-1}{4}+x$, which we can easily adapt to any form of an equation of the 5th degree, thus

$$
\begin{array}{lll}
x^5-x^4-x^3-x^2-x^5 & = 120\binom{\mathrm{T}x-2}{6}+ & 30\binom{\mathrm{T}x-1}{4} \quad +x \\
-x^4 & = & -24\binom{\mathrm{T}x-2}{5}-18\binom{\mathrm{T}x-1}{4}-2\,{}^{\mathrm{T}x}_{3}+x \\
-x^3 & = & -6\binom{\mathrm{T}x-1}{4} \quad -x \\
-x^2 & = & -2\,{}^{\mathrm{T}x}_{3}+x \\
-x & = & -x \\
\hline
-x^4-x^3-x^2-x & = & -24\binom{\mathrm{T}x-2}{5}-68\binom{\mathrm{T}x-1}{4}-4\,{}^{\mathrm{T}x}_{3} \\
\hline
\text{whence } x^5-x^4-x^3-x^2-x & = 120\binom{\mathrm{T}x-2}{6}-24\binom{\mathrm{T}x-2}{5}-10\binom{\mathrm{T}x-1}{4}-4\,{}^{\mathrm{T}x}_{3}+x &
\end{array}
$$

Thus, let $x=5$ then $3125-625-125-25-5=2345=120\times 21-24\times 15+12\times 20-4\times 15+5=2520-360+240-60+5=2345$.

6th Powers.

We obtain by the analysis of a series of 6th powers the following formula of their formation, which is also different from that derived from the synthesis.

$$x^6=720\binom{\mathrm{T}x-3}{7}+360\binom{\mathrm{T}x-2}{6}+120\binom{\mathrm{T}x-2}{5}+60\binom{\mathrm{T}x-1}{4}+2\binom{\mathrm{T}x-x}{3}.$$

which we may adapt to any form of an equation of the 6th degree.

For instance, let the value of x be required in this equation.

$$x^6-3x^5+6x^4-11x^3=77861.$$

$$1.\quad x^6 = 720\binom{Tx-3}{7} + 360\binom{Tx-2}{6} + 120\binom{Tx-2}{5} + 60\binom{Tx-1}{4} + 2\,{}^{Tx}_{3} - 6x$$

$$2.\quad +6x^4 = \qquad\qquad 6.24\binom{Tx-2}{5} + 6.12\binom{Tx-1}{4} + 6.2\,{}^{Tx}_{3} - 6x$$

$$\text{Sum},\quad = 720\binom{Tx-3}{7} + 360\binom{Tx-2}{6} + 264\binom{Tx-2}{5} + 132\binom{Tx-1}{4} + 14\,{}^{Tx}_{3} - 7x$$

$$3.\quad -3x^5 = \qquad -3.120\binom{Tx-2}{6} \qquad -3.30\binom{Tx-4}{4} \qquad + 3x$$

$$4.\quad -11x^3 = \qquad\qquad -11.6\binom{Tx-1}{4} \qquad +11x$$

$$\text{Sum}\quad -3x^5 - 11x^3 = \quad -360\binom{Tx-2}{6} \qquad -156\binom{Tx-1}{4} \qquad +14x$$

$$\left.\begin{matrix} x^6 - 3x^5 + 6x^4 \\ -11x^3 \end{matrix}\right\} = 720\binom{Tx-3}{7} \qquad +264\binom{Tx-2}{5} - 24\binom{Tx-1}{4} + 14\,{}^{Tx-1}_{3} - 21x$$

$$= 77861.$$

Dividing 77861 by 720, we obtain 108; looking in the table of numbers of the 7th order for the nearest lower number, we find it to be 84, the 4th term, so $4 = x-3$, and $4+3 = x = 7$, and

$$720\times 84 + 264\times 70 - 24\times 56 + 14\times 28 - 21\times 7 = 77861$$
$$= 60480 + 18480 - 1344 + 392 - 147 = 77861.$$

Theorems for 7th Powers.

The analysis of a series of 7th powers leads to the following formula of their formation:

$$x^7 = 5040\binom{Tx-3}{8} + 1680\binom{Tx-2}{6} + 126\binom{Tx-1}{4} + x.$$

8th Powers.

We find in the same way this formula for a series of 8th powers.

$$x^8 = 40320\binom{T\,x-4}{9} + 20160\binom{T\,x-3}{8} + 10080\binom{T\,x-3}{7}$$
$$+ 5040\binom{T\,x-2}{6} + 504\binom{T\,x-2}{5} + 252\binom{Tx-1}{4} +$$
$$2\binom{T\,x-1}{3} + x.$$

I subjoin an example of the application of this formula to an equation of the 8th degree.

$$x^8 + 35x^5 - 13x^3 + 7x^2 + 151x = 7025393157$$

The coefficient of the 1st term of the formula being 40320, we divide by 40,000, and obtain 170,000; the nearest lower number of the 9th order is 125,970, the 13th term; and since the 1st term of the formula, or of the 9th order, corresponds to $x-4$, we have $13 = x-4$, and $x = 17$.

We shall prove this.

$$x^8 = 40320\,{}^{Tx-4}_{9} + 20160\,{}^{Tx-3}_{8} + 10080\,{}^{Tx-3}_{7} + 5040\,{}^{Tx-2}_{6} + 504\,{}^{Tx-2}_{5} + 252\,{}^{Tx-1}_{4} + 2\,{}^{Tx-1}_{3} + x$$

$$+35x^5 = \qquad\qquad 35.120\,{}^{Tx-2}_{6} \qquad + 35.30\,{}^{Tx-1}_{4} + 35x$$

$$-13x^3 = \qquad\qquad\qquad\qquad - 13.6\,{}^{Tx-1}_{4} - 13x$$

$$+7x^2 \qquad\qquad\qquad\qquad = + 7.2\,{}^{Tx-1}_{3} + 7x$$

$$+151x \qquad\qquad\qquad\qquad\qquad\qquad + 151x$$

$$\left.\begin{matrix} x^8+35x^5-13x^3 \\ \times 7x^2+151x \end{matrix}\right\} = 40320\,{}^{Tx-4}_{9} + 20160\,{}^{Tx-3}_{8} + 10080\,{}^{Tx-3}_{7} + 5040 + 4200\,{}^{Tx-2}_{6} + 504\,{}^{Tx-2}_{5}$$

$$+\left(252+1050-78\right){}^{Tx-1}_{4} + \left(2+14\right){}^{Tx-1}_{3} + (1+35-13+7+151)x.$$

$x=17$	$x-4=13$	13th term 9th order =	125970 ×	40320	=	5079110400
	$x-3=14$	14th term 8th order =	77520 ×	20160	=	1562803200
		14th term 7th order =	27132 ×	10080	=	273490560
	$x-2=15$	15th term 6th order =	11628 ×	5040	= 58605120	
			×	4200	= 48837600	
						107442720
		15th term 5th order =	3060 ×	504	=	1542240
	$x-1=16$	16th term 4th order =	816 ×	252	= 205632	
			×	1050	= 856800	
					1062432	
			×	−78	= −63648	
						998784
		16th term 3d order =	136 ×	2	= 272	
			×	14	= 1904	
						2176
	$x=17$	$=(17+595-221+119+2567)=181$ ×		17	=	3077
		$x^8+35x^5-13x^3+7x^2+151x =$				7025393157

If the same equation had been made equal to 779.76852441, where there are 8 decimals in the power, we should have known that there was one decimal place in the root.

Dividing $\frac{779.76852441}{40000}$ we have 1,900,000 where the nearest lower number of the 9th order is 1,562275, the 19th term, which being put equal to $x-4$ we should have $23 = x$; but we may easily perceive that this is too great, for if x were $2 +$ a decimal fraction, its 5th power would be $32 \times 35 = 1120$, which, independent of the other terms, exceeds the limits of the equation. The integral part of the root therefore, cannot exceed unity, and there being only one decimal place in the root, we may begin with the highest digit, and make $x = 1.9$. and since $.35\ x^5 = 35.120\ ^{\mathrm{T}}\frac{x-2}{6}$ we take the $19-2 = 17$th term of the 6th order $= 20349 \times 4200 = 1936.75200$, and cutting off 5 decimal places, we have 1936 for the value of this term only, which we observe exceeds the limits of the equation. We need not try 1.8, because the last decimal place being 1, cannot proceed from an even number; we therefore try 7, and making $x = 1.7$, we shall find it succeed as before, attending only to the due placing of the decimal points, according to the powers to which the terms belong.

Thus, taking the terms of x^8:

$40320\,\mathrm{T}\frac{x-4}{9}$ equal, as before, to 5079110400, and allowing 8 places of decimals — 50.79110400

$20160\,\mathrm{T}\frac{x-3}{8}$ = allowing also 8 decimal places = 15.62803200

$10080\,\mathrm{T}\frac{x-3}{7}$ = allowing 8 decimal places = 2.73490560

$5040\,\mathrm{T}\frac{x-2}{6}$ = allowing 8 decimal places = .58605120

$504\,\mathrm{T}\frac{x-2}{5}$ = allowing 8 decimal places = .01542240

$252\,\mathrm{T}\frac{x-1}{4}$ = allowing 8 decimal places = .00205632

$2\,\mathrm{T}\frac{x-1}{3}$ = allowing 8 decimal places = .00000272

x = allowing 8 decimal places = .00000017

Taking next the terms of $35\,x^5$ with 5 decimal places.

$4200\,\mathrm{T}\frac{x-2}{6}$ = allowing 5 decimal places = 488.37600

$1050\,\mathrm{T}\frac{x-1}{4}$ = allowing 5 decimal places = 8.56800

$35x$ = allowing 5 decimal places 0.00595

Taking now the terms of the 2d power.

$14\,\mathrm{T}\frac{x-1}{3}$ = allowing 2 decimal places 19.04

$7x$ = allowing 2 decimal places 1.19

$151x$ = allowing 1 decimal place 256.7

Sum of the positive terms of the equation = 843.63752441

Sum of the positive terms of the equation = 843.63752441
From which there is to be subtracted $-13\,x^3$

$78\,\mathrm{T}\frac{x-1}{4}$ allowing 3 decimal places = 63.648

also $13\,x$ allowing 3 decimal places = 221

63.869

Remains $x^8+35\,x^5-13\,x^3+x^2+151\,x^1 =$ 779.76852441

9th Powers.

$$x^9 = 362880\left(\mathrm{T}\frac{x-5}{10}\right) + 151200\left(\mathrm{T}\frac{x-3}{8}\right) + 17640\left(\mathrm{T}\frac{x-2}{6}\right) + 510\left(\mathrm{T}\frac{x-1}{4}\right) + .$$

10th Powers.

$$x^{10} = 3628800\left(\mathrm{T}\frac{x-5}{11}\right) + 1814400\left(\mathrm{T}\frac{x-4}{10}\right) + 1209600\left(\mathrm{T}\frac{x-4}{9}\right) + 604800\left(\mathrm{T}\frac{x-3}{8}\right) + 105840\left(\mathrm{T}\frac{x-2}{7}\right) + 52920\left(\mathrm{T}\frac{x-2}{6}\right) + 2040\left(\mathrm{T}\frac{x-2}{5}\right) + 1020\left(\mathrm{T}\frac{x-1}{4}\right) + 2\left(\mathrm{T}\frac{x-1}{3}\right) + x.$$

I shall here conclude this summary view of a very tedious inquiry, by shewing the application of this formula for 10th powers, to the resolution of an equation of the 10th degree; but I shall not trouble the reader with the whole work, as it is

only tedious, without being attended with any other difficulty.

Let the equation be $x^{10} + 3x^9 + 6x^6 + 2x^3 + 57x = 7099413641225$.

Dividing this number by 4,000,000, for it is unnecessary to go nearly to work with such high powers, we obtain 1774353; looking in the table of the 11th order, we find the nearest lower term to be 1,144066 the 14th term; so we have $14 = x - 5$ and $x = 19$.

The numbers differ so very widely from each other, that there is no chance of mistaking the approximate root; and if the root is commensurate, as in the present case, we may ascertain it very easily, by attending to the last figure of the power, which in this case is 5.

Now, if we attend to the series of powers of the different orders, we shall find, that their endings are the same at the interval of every four powers.

Thus the series of 1st powers or natural numbers, the 5th powers, the 9th, the 13th, 17th, 21st, &c. powers, all end with the regular series of natural numbers.

The 1st, 5th, 9th, 13th, &c. end with	1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, &c.
The 2d, 6th, 10th, 14th, 18th, &c.	1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, &c.
The 3d, 7th, 11th, 13th, 19th, &c.	1, 8, 7, 4, 5, 6, 3, 2, 9, 0, 1
The 4th, 8th, 12th, 16th, 20th, &c.	1, 6, 1, 6, 5, 6, 1, 6, 1, 0, 1

Terms or roots,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1, &c.
The equation being x^{10} we have the last figures of the 10th powers × 1 =	1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, &c.
$+3x^9$ = last figures of 9th powers × 3 =	3, 6, 9, 2, 5, 8, 7, 4, 7, 0, 1
Sum	4, 0, 8, 8, 0, 4, 6, 8, 8, 0, 1

Terms	1, 2, 3, 4, 5, 6, 7, 8, 9
Endings of $x^{10}+3x^9=$	4, 0, 8, 8, 0, 4, 6, 8, 8
$+ 6x^6 =$ last figures of 6th powers, $\times 6$	6, 4, 6, 4, 0, 6, 4, 4, 6, 0
$+ 2x^3 =$ last figures of 3d powers, $\times 3 =$	2, 6, 4, 8, 0, 2, 6, 4, 8
$+ 57x =$ roots $\times 57=$	7, 4, 1, 8, 5, 2, 9, 6, 3, 0
Sums = last figures in the equation given,	9, 4, 9, 8, 5, 4, 9, 2, 5

we observe, therefore, that x can only be 5 or 9, which shews that 19, as already found, is the true root.

CHAPTER IV.

MISCELLANEOUS THEOREMS FOR THE RESOLUTION OF EQUATIONS.

1. *A General Theorem for the Resolution of Quadratic Equations.*

EVERY quadratic equation, being the product of two factors, these factors, if equal, will produce x^2, when the root is found immediately, by the formula $x^2 = 2^{\frac{Tx-1}{3}} + x$.

But if the factors are not equal, one of them is the greater, and calling the lesser factor x, we have $x+d$ for the greater factor, and multiplying $x+d$ by x, we have $x^2+d\,x$, where d, the difference of the factors, $= p$ the coefficient of the 2d term in the general formula for a quadratic equation $x^2+p\,x=a$; or if we make x the greater factor, then $x-d$ will be the less, and multiplying we have $x^2-d\,x$, where d, the difference of the factors, $= p$ the coefficient of the 2d term, in the general formula, $x^2-p\,x=a$.

Where d, the difference of the factors, is greater than x, then $d=x+d'$, and if the 2d term of the equation is negative, then we have $x\times(x-(x+d')=x^2-x^2-d'x-=d'x=-a$, where d' is one

of the factors, and comparing this with $x^2 - px = -a$, we have $x^2 - px = -d'x$, and transposing $-px = -x^2 - d'x$, and dividing by x, $-p = -x - d'$, or changing the signs $p = x + d'$, where p, the coefficient of the 2d term, is equal to the sum of the factors $x + d'$, this theorem may therefore be expressed thus,

In the formula for quadratic equations, $x^2 \pm p x = \pm a$, *when* a *is affirmative,* p *is equal to the difference of the two roots of the equation. And where* a *is negative,* p *is the sum of the two roots.*

It may be remarked, that this theorem is just a particular case of a more general theorem, that the coefficient of the 2d term of any regular equation, is the sum or difference of the roots of the equation.

We shall give examples of these three cases.

Given $x^2 + 7x = 144$. Here 7 is the difference of the factors of 144; supposing $x = 10$, we have $x + d = 10 + 7$, and $10 \times 17 = 170$, which is too great; our next factors are $9 \times 16 = 144$, so $x = 9$ or -16.

Given $2x^2 + 3x = 275 = x^2 + \frac{3}{2}x = \frac{275}{2} = 137\frac{1}{2}$. Supposing $x = 10$, we have $10 + \frac{3}{2} = 11\frac{1}{2}$ for the other factor, $10 \times 11\frac{1}{2} = 115$, which is too small; making $x = 11$, we have $11 \times 12\frac{1}{2} = 132 + 5\frac{1}{2} = 137\frac{1}{2}$, so $x = 11$ or $-12\frac{1}{2}$. Thus if $x = 11$, $2x^2 + 3x = 242 + 33 = 275$; and if $x = -12\frac{1}{2}$, $2x^2 = 144 + 12 + \frac{1}{2} = 150\frac{1}{2} \times 2 = 312\frac{1}{2} - 3x = -3 \times 12\frac{1}{2} = -37\frac{1}{2}$, and $312\frac{1}{2} - 37\frac{1}{2} = 275$.

2d Case.

Given $x^2 - 4x = 32$. Supposing $x = 10$, then $10 \times (10 - 4) = 10 \times 6 = 60$, which is too great, 9×5 is also too great; but 8×4 answers the conditions, and $x = 8$, or -4, if $x = 8$, then $8^2 = 64 - 32 = 32$; and if $x = -4$, then $x^2 = 16$, and $-4x = -4 \times -4 = +16$, and $x^2 - 4x = 16 + 16 = 32$.

3d Case.

Given $x^2-11x=-24$; here $x+d\ 11=$ and the two factors are 1×10, 2×9, or 3×8. This last succeeds, and shews us that $x=+3$, or $+8$.

If $x=3$, then $x^2-11x=\ 9-33=-24$

or $x=8$ . . $64-88=-24$. So when the value of a quadratic equation, and the 2d term are both negative, both the roots are positive.

$x^2-13x=-22$. Here the sum of the factors is 13, and we have $2\times11=22$, which succeeds, and shews the roots to be 2 and 11. If $x=2$, then— $x^2-13x=4-\ 26=-22$

if $x=11$ $\quad 121-143=-22$.

This method for quadratic equations belongs to the following more general theorem.

Theorem for finding the Roots of Equations by means of the Coefficients of the 2d Term.

If an equation be perfect or regular, the coefficient of its first term will be unity, that of its 2d term will be the sum of all its roots, or factors; the coefficient of the 3d term will be the sum of the products of each factor into every other factor, two and two; the coefficient of its 4th term will be the sum of the products of its factors, three and three, and so on, until its last term, where the coefficient will be the product of the whole roots or factors. So that the coefficient of the 2d term is always the sum, and the last term the product of all the factors.*

But if the sum and product of any number of factors be given, it is easy to find the factors; for if we divide the product of

* When the roots are fractions, this law of the coefficients is reversed.

all the factors continually, by its least divisors, until it is exhausted, we shall obtain all the divisors; if we combine these into two compound factors, and add them, we can compare the sum of the two compound factors with the coefficient of the 2d term, and observe the difference; then subdividing either or both of the compound factors, we observe the difference between the sum and product of these 2d factors, which we deduct from the difference of the sum and product of the first; subdividing the factors continually, until we obtain as many factors as there are roots in the equation, and deducting the successive differences from the first difference, until it is exhausted. An example will make this more clear.

Let the coefficient of the 2d term of any regular equation of the 2d degree be 12, and the coefficient of the last term be 36, then dividing continually we have $\frac{36}{2}, \frac{18}{2}, \frac{9}{3}, \frac{3}{3}, = 1$, and we observe that 2, 2, 3, 3, are its whole factors, excepting 1 and 36. These we may combine into $(3 \times 2) + (3 \times 2) = 6$. But since $6 + 6 = 12$, and $6 \times 6 = 36$, both roots of the equation are equal. We may observe that, although 4×9, 3×12, 2×18, and 1×36, are all equal to 36, that the sum of all these other pairs of factors exceed 12, for where the factors are equal, their sum is the least possible. We observe also, that the sums of all the factors are also different, those nearest equality having the lesser sum, which rule is quite general, and may be extended to any number of factors; so that there can never be two sets of factors having their sums equal, unless one factor in the one set, shall be as much less than the corresponding factor in the other set, as some other of the factors of the same set shall exceed some other corresponding factor in the other set, which coincidence can very rarely happen.

Let there be three factors, whose sum is 17, and product 72,

here we find the whole divisors of 72 to be, 2, 2, 2, 2, 3, 3; we have 2, 2, 2, 2, 3, 3, which we may combine into 8×9; here $8+9=17$, which is equal to the sum of the three roots in the equation, so that the 3d factor behoved to be 0, which shews us that we have assumed factors too nearly equal. Taking therefore the factors which have the next least difference, these are 6 and 12, the sum of which being 18, exceeds the sum of the three unknown factors by 1, so we subdivide 6, the lesser factor into 2×3, where $2\times3=6$, and $2+3=5$, the difference being 1, exhausts the first difference, and we have therefore 2, 3, 12, for the three factors, the sum of which is 17, and their product 72.

Let there be 3 factors whose sum is 21, and product 231; dividing 231 by 3, we obtain 77, and dividing 77 we obtain 3, 7, 11, for the three factors, the sum of which is 21, and product 231; if we had combined these into 21×11, the sum of these factors being 32, exceeds 21 by 11; subdividing therefore, 21 into 7×3, we observe that $7\times3-7+3=21-10=11$, which exhausts the 1st difference.

Or, if we had taken 7×33 for the compound factors, their sum is $40=21+19$; subdividing 33 into 3 and 11, we have $3\times11-(3+11)=33-14=19$.

This method, however, can only be applied to a regular equation; and even where the equation is regular, there are several cases to be distinguished, viz. where one or more of the roots are negative; but it is unnecessary to go into this inquiry, as the following method is still more convenient, and is applicable to every complete equation, when the roots are rational and commensurate.

II.

A General Method of Resolving Equations, by finding the sum of the Squares of all the Roots.

1. In the case of cubic equations, and where all the roots are positive, the equation is of this form;

$$x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr=0,$$

where there is no difficulty in applying the method of the divisors of the last term; but we may also, in this case, obtain the roots differently: Thus, if we square the coefficient of the 2d term $=-(p+q+r)^2$, we obtain $p^2+q^2+r^2+2pq+2pr+2qr$

and subtracting twice the coefficient of the 3d term $=pq+pr+qr=\times 2=$ $2pq+2pr+2qr$

There is left the squares of the roots $p^2+q^2+r^2$ * * *

2. In cubic equations, where there is one root of the equation negative, the equation is of this form:

$$x^3-(p+q-r)x^2+(pq-pr-qr)x+prq=0.$$

If we take the square of the coefficient of the 2d term of this equation, we have

$p^2+q^2+r^2+2pq-2pr-2qr$, and subtracting twice the coefficient of the 3d term $=$ $2pq-2pr-2qr$

There is left, as before, the sum of the squares of the roots $p^2+q^2+r^2$ * * *

The case is precisely similar where there are two negative roots; and where the roots are all negative, the resolution is the same as in the 1st case, where the roots are positive.

Examples.

1. $x^3-18x^2+95x-126=0$.

Here $18^2 = 324$
$-95 \times 2 = 190$

Remains $134=p^2+q^2+r^2$; the highest square might be 11, as $121+4+1=126$; but 11 is not a divisor of 126, neither is 10, though $100+25+9=134$; trying 9, we have $81+49+4=134$; so the true roots are 9.7.2, which is proved by their sum being equal to 18, the coefficient of the 2d term, and that their product is equal to 126.

Making one of these roots, as 2, negative, we have this equation from the same roots,

$$x^3-14x^2+31x+126=0,$$

and $14^2 = 196$
$-31\times 2 = 62$

Leaves $134 = p^2+q^2+r^2$, as before.

Supposing two negative roots, as 2 and 7, in the same example, we shall have for the coefficient of the 2d term
$-2-7+9=0$, and for the

coefficient of the 3d term, $\left.\begin{matrix}-2\times-7=+14\times 2\\ -2\times+9=-18\times 2\\ -7\times+9=-63\times 2\end{matrix}\right\}=-134=$
$=2pq-2pr-2qr=$
$=-p^2-q^2-r^2$

from which we obtain this theorem: That where the greatest of three factors is equal to the sum of the other two, that twice the product of the greatest factor into the sum of the other

two, exceeds twice the product of the other factors into each other, by the sum of the squares of all the three factors; which indeed follows as a corollary from this proposition: That the square of a number is equal to the squares of the parts into which it is divided, together with twice the product of the parts.

We observe, likewise, that where the sum of the two negative roots is equal to the positive root, there cannot be a second term to the equation.

This method applies to regular and rational equations of *every degree*, as the coefficient of the *2d term* is always the sum, and the coefficient of the *3d term* is always the product of all the roots, taken two and two, into each other, whatever be the number of roots.

III.

A particular Theorem for the Solution of Cubic Equations.

$$x^3 = 6m\,x^2 - (12\,m^2 - 1)x + 48\binom{\mathrm{T}m-1}{4} + 6\,m.$$

Where m is always an integer, and the nearest integer less than, or equal to, one half of the approximate value of x.

Thus if $x=1$	$m=1$	$x=4$	$m=2$	$x=8$	$m=4$
$x=2$	$m=1$	$x=5$	$m=2$	$x=9$	$m=4$
$x=3$	$m=1$	$x=6$	$m=3$	$x=10$	$m=5$
		$x=7$	$m=3$	$x=11$	$m=5$, &c.

which theorem depends upon this proposition, that the cubes are equal to the differences of the squares of the trigonal numbers, which it is easy to demonstrate.

For since $x^2 = \binom{\mathrm{T}x}{3} + \binom{\mathrm{T}x-1}{3}$

and $x = \binom{\mathrm{T}x}{3} - \binom{\mathrm{T}x-1}{3}$

$x^2 \times x = x^3 = \left(\binom{\mathrm{T}x}{3} + \binom{\mathrm{T}x-1}{3}\right)\left(\binom{\mathrm{T}x}{3} - \binom{\mathrm{T}x-1}{3}\right)$ but as the sum of any two quantities multiplied into their difference is equal to the difference of their squares, therefore $x^3 = \binom{\mathrm{T}x}{3}^2 - \binom{\mathrm{T}x-1}{3}^2$.

And by substituting a series of values of x in regular arithmetical progression, and comparing the cubes, expressed in this way in terms of the differences of the squares of trigonal numbers, with the formula, $x^3 = 6\binom{\mathrm{T}x-1}{4} + x$, I arrived at the theorem stated above.

An example will shew more clearly the application of this formula. Let $m=3$, so that the value of x is between 6 and 7.

Then $6\,m\,x^2 = (6.3)x^2 = 18\,x^2$

$(12\,m^2-1)x = 12.9 = 108-1 = 107\,x$

$48\binom{\mathrm{T}m-1}{4} = 48\times 4 = 192$

$6\,m = 18$

So if the equation be $x^3+4x=240$

Then $18x^2-107x+4x+192+18=240$

$= 18x^2-103x+210=240$

$= 18x^2-103x=240-210=30$

a quadratic equation which we may resolve in the usual way, and we shall obtain $x=6$.

But if decimals are employed, m becomes one-half of the whole expression of the value of x taken as an integer: thus, if $x=1.5$, $m=7$, if $x=1.36=m=68$, which renders this theorem of less value; although we may always know when m has been assumed wrong, and what is the necessary correction. For where the root is an even number, if m be assumed 1, 2, 3, 4, 5, 6, 7, 8, &c. too great or too small, the result will be greater or less than the truth by the corresponding term of this series, 6, 60, 210, 504, 990, 1716, &c. being a series of $48\binom{n-1}{4}+6\,n$, where n is the error or difference between the assumed value of m, and the nearest integer to $\frac{1}{2}\,x$.

And where the root is an odd number, the corresponding series will be 24, 120, 336, 720, 1320, 2184, being the series of $48\binom{n-1}{4}+24\binom{n-1}{3}+24\,n$.

I now proceed to the statement of some more general and easy methods of resolving equations.

IV.

A New Method for the Resolution of Equations.

It would seem to have been held as a fundamental axiom and first principle in algebraical analysis, that it was necessary, in order to the easier resolution of equations, to transfer all the known quantities to one side of the equation, and the unknown quantities to the other side, that, by the comparison of things known, with things unknown, the value of the latter might be obtained.

But this principle has perhaps been pushed too far; for it will be found, that great advantage is frequently derived from placing the highest term of the unknown quantity to the other side of the equation, so as to compare it taken singly, and without affection, with the known quantity, or by dividing the known quantity by the unknown quantity or its powers, in which case the unknown quantity is so far ascertained, as being a divisor of the known quantity; for the quantity itself being known, its divisors are given,—and we may have as many conditions of this kind as there are degrees in the equation.

This will be best illustrated by examples.

1st Method, by transferring the highest powers of the unknown quantity to the other side of the equation.

I shall extract my examples of the application of this method from Euler's Algebra, 1st English edition, p. 415 *et sequen.*

Examples of the 1st Method.

1. $x^3=6x+9.—x^3-6x=9$. Changing the signs and trans-

posing, we obtain $6x=x^3-9$; and taking for x^3 the nearest cube above 9, we have $6x=27-9=18$, so $x=3$.

2. $x^3=3x+2$. $x^3-3x=2$. Changing the signs and transposing, we have $3x=x^3-2$; and taking $x^3=$ to the nearest cube above 2, we have $x=2$.

3. $x^3=6x+40$. $x^3-6x=40$. $6x=x^3-40$, and making $x^3=$the nearest cube above $40=64$, we have $x=4$.

4. $x^3-6x^2+11x^1=6$. $6x^2-11x=x^3-6$ and taking $x^3=8$ we have $x=2$.

5. $x^3-6x^2+13x-12=0$ $6x^2-13x=x^3-12=27-12$, and $x=3$.

6. $x^4=6x+4$, which Euler says is perfectly irreducible, and cannot be expressed otherwise than by $\sqrt[3]{2+2\sqrt{-1}}+\sqrt[3]{2-2\sqrt{-1}}$. But this equation $x^3=6x+4$, gives, according to the preceding rule,

$x^3-6x=4$, and changing the signs and transposing x^3 we obtain $6x=x^3-4$, and assuming $x^3=8$, we have $6x=8-4=4$ and $x=\frac{4}{3}$ which does not answer. But if we assume x negative, then $x^3=-x^3$ and $6x=-6x$, in which case our equation becomes $-x^3=-6x+4$, transposing $-6x=-x^3-4$ and taking the nearest cube above 4 we have $-6x=-8-4$ and $x=-2$.*

This method is applicable also to equations of the higher degrees. But the higher equations can always be reduced to lower terms by the following method, which also affords a very easy and obvious resolution.

* The irrational expressions obtained by Cardan's rule, $x=\sqrt[3]{2+2\sqrt{-1}}+\sqrt[3]{2-2\sqrt{-1}}$ are equal to $-1+\sqrt{-1}$ $-1-\sqrt{-1}=-2$; for if we cube these expressions, we shall obtain $2+2\sqrt{-1}+2-2\sqrt{-1}$.

2d Method,—Dividing by the unknown quantity.

Let the equation be $x^3 \pm px^2 \pm qx = a$; dividing by x we have $x^2 \pm px \pm q = \frac{a}{x}$, here x is found as one of the divisors of a.

Dividing again by x, we obtain $x \pm p \pm \frac{q}{x} = \frac{a}{x^2}$; so that if a is not divisible by x^2, neither is $\frac{q}{x}$ divisible without a remainder. And if actually divided, the remainder of the one division is equal to the fractional remainder of the other.

For if q be greater than x, the rational part of the improper fraction is to be subtracted.

If the equation be of the 4th degree, $x^4 \pm p x^3 \pm qx^2 + rx = a$, dividing by x, we have $x^3 \pm px^2 \pm qx + r = \frac{a}{x}$; here x is found as a divisor of a. Dividing again by x, we have $x^2 \pm px \pm q \pm \frac{r}{x} = \frac{a}{x^2}$. Here, if a is divisible without a remainder by x^2, $\frac{r}{x}$ is so also without remainder. If not, the remainder of $\frac{a}{x^2}$ will be equal to the fractional remainder of $\frac{r}{x}$. And in this way we may obtain as many equations of this kind as there are dimensions in the equation to be resolved.

I subjoin a few examples of the application of this method to the solution of cubic equations, some of which have been taken from Le Caille's *Cours Elementaire de Mathematiques.*

Given, $x^3 - 39x = 70$

$x^2 - 39 = \frac{70}{x}$ wherefore x must be a divisor of 70; trying therefore the divisors of 70,

We have $\frac{70}{2}=\frac{35}{5}=7$, where the only divisors of 70 are 2, 5, 7, which taken plus or minus, must be the three roots; assuming $x=2$, we have $8-39\times 2=70$, which does not answer, but taking -2 we have $-8+78=70$, so -2 is one of the roots.

Taking $5-x=$ we have $125-195=70$, which if taken positive does not answer, but being taken negative we have $-125+195=70$, and -5 is another root. Finally,

Taking $x=7$, we have $343-39\times 7=-273=70$, so 7 is the 3d root.

Required x in this equation.

$$x^3-17x-4=0$$
$$x^3-17x=4$$
$$x^2-17=\frac{4}{x}$$ where x must be a divisor of 4, so $x=1$ or $x=2$, or $x=4$; taking $x=1$ we have $1-17=4$ which does not answer.

Making $x=2$ we have $4-17=2$, which does not answer; trying $x=4$ we have $16-17=1$, which does not answer taken positive, but, taken negative, we obtain $16-17=-1=\frac{4}{-4}$ whence -4 is a real root, from which the other roots may be found.

Given $x^3-3x=18$, dividing by x,

$x^2-3=\frac{18}{x}$; wherefore $\frac{18}{x}+3=x^2$, that is, $\frac{18}{x}+3$ must be a square number.

The divisors of 18 are 1, 2, 3, 6, 9, 18;

Adding 3 . . . 3, 3, 3, 3, 3, 3

We have . . . 4, 5, 6, 9, 12, 21; so that x can only be

18, or 3, since the quotients 1 and 6, increased by 3, are the only squares; we do not try 18, as it evidently will not succeed; but making $x=3$, we have $27-9=18$.

Or dividing the same equation by x^2, we shall have $x-\frac{3}{x}=\frac{18}{x^2}$, the only square which is a divisor of 18 excepting 1, is 9, so we have $3-\frac{3}{3}=\frac{18}{9}=2$.

Finally, dividing by x^3, we have

$1-\frac{3}{x^2}=\frac{18}{x^3}$, whence it is evident $\frac{18}{x^3}$ must be a fraction. Taking the nearest cube greater than 18, we obtain $1-\frac{3}{9}=\frac{18}{27}$ $=1-\frac{1}{3}=\frac{2}{3}$; so $x=3$.

Given $x^3-109x+420=0$

$=x^3-109x=-420$, dividing by x

we obtain $x^2-109=-\frac{420}{x}$.

So that we must take x^2 less than 109.

$100-109=-9$, which is not a divisor of 420.

$81-109=-28$, which, though a divisor of 420, does not answer.

$64-109=-45$ is not a divisor.

$49-109=-60$, is a divisor, and gives 7 for one of the roots.

$36-109=-73$, not a divisor.

$25-109=-84$, is a divisor, and gives 5, another root.

If we desire to obtain the 3d root, we must arrange our equation differently, thus,

$$x^3=109x+420$$

$$x^2=109+\frac{420}{x}$$

$x^2-109=\frac{420}{x}=0$; taking therefore the squares above 109, we have

$121-109=\frac{420}{11}=0$, which does not answer; but

$144 - 109 = \frac{420}{12} = 35$ succeeds, and shews the other root to be 12.

The roots, therefore, are 5, 7, and 12.

We should have found the middle root directly by dividing by x^3;

for $x^3 - 109x + 420 \div x^3 = 1 - \frac{109}{x^2} + \frac{420}{x^3}$, and taking the nearest cube to 420, we have $1 - \frac{109}{49} + \frac{420}{343} = 1 - 2 - \frac{11}{49} + 1 + \frac{11}{49} = 0$. So $x = 7$.

It has been objected to this method, that it does not apply where the differences of the roots are great, as in the equation $x^3 - 27x^2 + 179x - 153 = 0$, where the nearest cube to 153 being 125, gives 5 for a root, which it is not in this equation.

Also, in this equation $x^3 - 54x^2 + 203x - 150 = 0$, we should obtain 5 for a root of the equation, which it is not.

In both these examples, however, unity is one of the roots or factors of the equation; and where unity is a factor of a cubic equation, it becomes of this form, $x^3 - (p+q+1)x^2 + (pq+p+q)x - pq = 0$; or if the root be -1, the equation is of this form, $x^3 - (p+q-1)x^2 + (pq-p+q)x - pq = 0$. It is easy, therefore, to resolve a cubic equation where unity is a root or factor. We have only to consider the last term of the equation as the product of two factors, whose sum is equal to the coefficient of the 2d term ± 1, and that the difference between the coefficient of the 3d term and the last term is also equal to the coefficient of the 2d term ± 1.

Thus, in the 1st of the preceding examples, $153 = 9 \times 17$, and $9 + 17 = 26 = 27 - 1$; also $179 - 153 = 26$, so that the roots are 1, 9, 17.

In the 2d equation, $150=3\times50$, and $50+3=54-1$; also $203-150=53=54-1$, so the roots are 1, 3, 50.

But where unity is not a factor of the equation, we certainly cannot apply the method directly. We shall, however, always be able, where the roots are commensurable, to find them without any trouble; for where the difference of the roots is great, the least factor will be a small number, and therefore one of the simple factors of the last term; and it will also be very near the factor which the nearest cube would point out. Indeed, if the factors were supposed to recede from the middle factor by an arithmetical progression, we should have $(x-1)\times x\times(x+1)=x^3-x$ $(x-2)\times x\times(x+2)=x^3-4x$ $(x-3)\times x\times(x+3)=x^3-9x$, &c. decreasing by the series of squares $\times x$; so that when the coefficient of x exceeds the difference of the cubes of x and $x-1$, that is, when it exceeds 6 times the trigonal of x, the method of the nearest cube will not answer, though it can rarely be far from the truth.

I proceed, therefore, to shew the application of this method, to finding the roots of an incommensurate equation.

Required to find x from the equation $x^3+7x=50$.

Here the nearest cube being 27, we have $x=3$, and $x^3+7x=27+21=48=50-2$.

The root is therefore greater than 3, and less than 4, that is, it is 3 and a fraction, we may represent this fraction by x', and we shall have

$(x'+3)^3+7x'+21=50$, where $(x'+3)^3+7x'=29$; cubing this quantity, we have $(x'^3+9x'^2+27x+27)+7x'=29$, whence

$$x'^3+9x'^2+34x=29-27=2.$$

Now, since x' is a fraction, its cube and square are small in proportion to its simple power, and if we neglect x'^3, we should have only to resolve the quadratic equation $9'x'^2+34x'$

$=2$, which might be easily effected by the formula in terms of the orders of numbers, by annexing cyphers to 2; but, without this, we may resolve it more simply by rejecting also the term $9x^3$, and we have this equation $34x'=2$, and $\frac{2}{34} = x'$; performing the division, we obtain .058 nearly $= x'$; so the root is $3 + .058 = 3.058$, and the cube of this quantity being $6.\binom{T.3057}{4} + 3058$, we have $(x'+3)^3$ . . . . .

4766078009 × 6 = 28596468054 + 3058 = 28.596471112

and $7(x'+3) = 3.058 \times 7 =$ 21.406

50.002471112;

and by substituting $(3058 - x'')$ for the new value of x, we should approximate still nearer to the root.

V.

General Method for the solution of Equations by Substitution of the Orders of ϕ, a Function of x.

We have formerly had occasion to notice, that, in the succession of powers of the series of natural numbers, there is an apparent separation of the powers into two parts, a principal root or stem, and an accessary root; which accessary root again separates into two parts, one part being incorporated with the principal stem, and a new accessary root separated, as the powers continue to advance in degree.

This is very perceptible in arithmetic, after every succession of four powers, for the 1st, 5th, 9th, 13th, 17th, 21st, &c. powers, all end with the same digit, being the final digit of the natural number or root; and in like manner, the 2d, 6th, 10th, 14th, &c. powers all end with the same digit, being the final digit of the square of the root. The 3d, 7th, 11th, 15th, &c. powers, all end with the digit of the cube of the root; and the 4th, 8th, 12th, 16th, &c. powers, end with the digit of the biquadrate of the root; and at the 5th power, the same series of the natural numbers begins and runs the same progress.

1. But we may easily demonstrate this law algebraically, if we make the fifth powers where the accessory root arises, $= F + x$, denoting by F the principal stem or root, and by x the accessory root, we shall have $x^5 = F + x$, and consequently multiplying always by x, we obtain,

$$x^6 = F x + x^2$$
$$x^7 = F x^2 + x^3$$

$x^8 = Fx^3 + x^4$

$x^9 = Fx^4 + x^5 = Fx^4 + F + x$, where the 2d accessory root appears; we have, therefore,

$$x^{10} = Fx^5 + Fx + x^2 = F \times (F + x) + Fx + x^2 =$$
$$= F^2 + 2Fx + x^2 = (F + x)^2$$

which we know to be true; since if $x^5 = F + x$, $(x^5)^2$ must be equal to $(F + x)^2$, $x^{15} = (F + x)^3$ &c,

We might evidently make the same substitution at the 9th power, which we might make equal to $F + x$, the 13th power $= F + x$, &c.

2. But it is more material to observe, that this substitution is equally competent to us, at the 2d, 3d, and 4th powers, or 6th, 7th, and 8th powers, as at the 5th, 9th, and 13th powers.

For if we make $x^2 = \phi + x$, we shall have the value of ϕ in terms of the orders of numbers; for as x^2 has been shewn equal to $2\left(\begin{smallmatrix}Tx-1\\3\end{smallmatrix}\right) + x$, we have $\phi = 2\left(\begin{smallmatrix}Tx-1\\3\end{smallmatrix}\right)$.

3. In the same way since $x^3 = 6\left(\begin{smallmatrix}Tx-1\\4\end{smallmatrix}\right) + x$, if we make $\phi_3 = x^3 - x$, we have $\phi_2 = 6\left(\begin{smallmatrix}Tx-1\\4\end{smallmatrix}\right)$.

But if ϕ_3 is made equal to $x^4 - x$, we shall not have the same expression of its value in terms of the orders of numbers which we formerly used for equations of the 4th degree; for we had, (p. 48.) $12\left(\begin{smallmatrix}Tx-1\\5\end{smallmatrix}\right) + 12\left(\begin{smallmatrix}Tx-2\\5\end{smallmatrix}\right) = x^4 - x^2$ in place of $x^4 - x = \phi_3$.

But since $x^4 - x^2 + (x^2 - x) = x^4 - x$, and as $x^2 - x = \phi$ (2.) we obtain $\phi_3 - \phi = x^4 - x = 12\left(\begin{smallmatrix}Tx-1\\5\end{smallmatrix}\right) + 12\left(\begin{smallmatrix}Tx-2\\5\end{smallmatrix}\right)$.

Multiplying by x, we have $\underset{3}{\varphi}x - \varphi x = \underset{3}{\phi}x - \underset{2}{\varphi} + \varphi = x^5 - x^3$; and adding $x^3 - x = \underset{2}{\varphi}$ to both sides, we have $\underset{3}{\varphi}x + \varphi = x^5 - x = \mathrm{F}$.

We have thus arrived at the 5th powers, or new series of the powers of numbers, where $x^5 = \mathrm{F} + x$, and F itself equal to (*vide* p. 60.) $120 \binom{\mathrm{T}x-2}{6} + 30 \binom{\mathrm{T}x-1}{4}$.

3. To propose, therefore, the substitution of these orders of φ, or functions of x; for the powers of x, in equations, would seem to be only another mode of proposing to resolve equations by means of the orders of numbers, as shewn in the preceding chapters; but we shall, upon farther consideration, be sensible of the advantage to be derived from this change in the notation, which will enable us not only to obtain the values of these different functions in terms of each other, but to subject them to all the ordinary operations of arithmetic and algebra.

This, however, is far too wide a field of inquiry for me to enter into at present; and as this method has only occurred in the course of writing out this memoir for the press, it is given only as a speculation for farther inquiry, and not as one which has been examined to any extent, or with the attention it appears to deserve.

Before we proceed farther, it is necessary to remark, that we are not limited in these substitutions to the particular functions of the powers of x less the root, which have been adopted only in the outset to shew the connection between that particular order of functions of x, and the factors we had employed in the preceding chapters in terms of the orders of numbers; although these will be found as convenient as any other, and we shall confine ourselves to them accordingly in this summary view of the subject: only it may be proper to take some notice of the figurate or polygonal numbers, which may be very con-

veniently applied in this way, as being perfectly regular in their formation, and unlimited in their extent.

Of the Polygonal or Figurate Numbers.

4. The polygonal or figurate numbers are all triangular, or numbers of the 3d order; or are the sums of an arithmetical series, where the difference of the terms of the series is equal to the number of the sides of the polygon less 2, with the addition of the root or side.

Thus triangular numbers $= 3-2=1\left(\frac{T-1}{3}\right)+x$

Squares or quadrangular numbers $= 4-2=2\left(\frac{Tx-1}{3}\right)+x$

Pentagonal numbers $= 5-2=3\left(\frac{Tx-1}{3}\right)+x$

Hexagonal numbers $= 6-2=4\left(\frac{Tx-1}{3}\right)+x$

and, generally,

m=gonal numbers $=m-2\left(\frac{Tx-1}{3}\right)+x$

6. These numbers may, therefore, be all found very readily in terms of the 3d order of numbers; thus, pentagonal numbers, for example,

	$=1, 5, 12, 22, 35, 51, 70,$ &c. $=$
3d Order	$=0, 1, 3, 6, 10, 15, 21,$ &c.
$\times 3$	$=0, 3, 9, 18, 30, 45, 63$
$+x$	$=1, 2, 3, 4, 5, 6, 7$
	$=1, 5, 12, 22, 35, 51, 70,$ &c.

7. But it may be curious to inquire also, for a single moment, into the laws of the formation of figurate numbers, by their peculiar factors.

The triangular numbers being equal to $\frac{x^2 \pm x}{2} = \frac{x}{2} \times (x \pm 1)$, shews that $\frac{x}{2} \times (x+1) =$ trigonal of x, and $\frac{x}{2} \times (x-1) =$ trigonal of $x-1$.

Thus, $3 \times \frac{3+1}{2} = 3 \times 2 = 6$, trigonal of 3,

$3 \times \frac{3-1}{2} = 3 \times 1 = 3$, trigonal of 2,

$5 \times \frac{5+1}{2} = 5 \times 3 = 15$, trigonal of 5,

$5 \times \frac{5-1}{2} = 5 \times 2 = 10$, trigonal of 4,

$7 \times \frac{7+1}{2} = 7 \times 4 = 28$, trigonal of 7,

$7 \times \frac{7-1}{2} = 7 \times 3 = 21$, trigonal of 6, &c.

8. Whence the trigonals of the odd numbers are equal to the odd numbers multiplied by the nearest integer greater than their half, and the trigonals of the even numbers are equal to the odd numbers multiplied by the nearest integer less than their half.

Wherefore, to find a trigonal root, we have only to find its two factors, one of which is double of the other, ± 1.

Thus, $21 = 7.3$, where 3 the least factor is less than $\frac{7}{2} = 7 \times \frac{7-1}{2} = 7 \times \frac{6}{2}$, and 6 is the root,

$28 = 7.4$, and 7 is the root, 4 being greater than $\frac{7}{2}$:

$36 = 9.4$, and $9-1 = 8$ is the root;

$45 = 9.5$, and 9 is the root, being $> \frac{9}{2}$;

$55=11\times5$, and $11-1=10$ is the root; since $5<\frac{11}{2}$

$66=11\times6$, and $6=\frac{11+1}{2}$; so 11 is the root.

9. Squares being equal to $2\left(\begin{smallmatrix}\mathrm{T}x-1\\3\end{smallmatrix}\right)+x$, or to $2\,{}^{\mathrm{T}x}_{3}-x$, and $\left(\begin{smallmatrix}\mathrm{T}x-1\\3\end{smallmatrix}\right)$ being equal to $\frac{x^2-x}{2}$ and ${}^{\mathrm{T}x}_{3}=\frac{x^2+x}{2}$, we have from either of these $x^2=\frac{2x^2}{2}=\left(\frac{x}{2}\times2x\right)$; whence we observe, that squares are the product of two factors, where the least factor is to the greater as 1 to 4. Thus,

$$\begin{aligned}1&=2\times\tfrac{1}{2}\\4&=4\times1\\9&=6\times1\tfrac{1}{2}\\16&=8\times2\\25&=10\times2\tfrac{1}{2}\\36&=12\times3,\ \&c.\end{aligned}$$

10. Pentagonal numbers $=3\left(\begin{smallmatrix}\mathrm{T}x-1\\3\end{smallmatrix}\right)+x=\frac{3x^2-3x}{2}+x=3(x-1)\times\frac{x}{2}$ or $=3x\times\frac{x-1}{2}$ whence we observe, that pentagonal numbers are the product of two factors, where the least factor is the nearest integer to $\frac{1}{6}$ of the greater factor.

$$\begin{aligned}1&=2\times\tfrac{1}{2}\\5&=5\times1\\12&=8\times1\tfrac{1}{2}\\22&=11\times2\\35&=14\times2\tfrac{1}{2}\\51&=17\times3\\70&=20\times3\tfrac{1}{2},\ \&c.\end{aligned}$$

Hexagonal numbers are equal to $4\binom{Tx-1}{3}+x$, and since $\binom{Tx-1}{3}=\frac{x^2-x}{2}$; $4\binom{Tx-1}{3}+x=\frac{4x^2-4x}{2}+x=\frac{4x^2-4x+2x}{2}$ $=\frac{4x^2-2x}{2}=(4x-2)\times\frac{x}{2}$, whence we learn that hexagonal numbers are equal to the product of two factors, where the less is the nearest integer, to $\frac{1}{7}$ of the other factor.

11. Thus the series of hexagonal numbers $=1, 6, 15, 28, 45, 66$, &c.

$$\begin{aligned}&=2\times\tfrac{1}{2}=1\\&=6\times 1=6\\&=10\times 1\tfrac{1}{2}=15\\&=14\times 2=28\\&-18\times 2\tfrac{1}{2}=45\\&=22\times 3=66, \text{ \&c.}\end{aligned}$$

12. It is needless to proceed to the other polygonal numbers, as the law is abundantly evident. We observe, 1*st*, That the greater factors are the successive terms of an arithmetical series beginning with unity, and whose common difference is $m-2$. And, 2*dly*, That the least factors are always formed of this series, $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4=\frac{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}{2}=\frac{x}{2}$.

Having shewn, therefore, how we may always find any figurate number from the root, and therefore inversely to find the root of any figurate number, we return to the investigation of the relations of the orders of φ, and of the figurate numbers, (which we may denote by f, f_2, f_3, making $f=$ the pentagonal numbers) proposed as functions of x, to be substituted in equations.

1*st*—*Of the Functions* φ.

13. 1. $\varphi=x^2-x$.

13. 2. $\varphi x = x^3 - x^2 = x^3 - \varphi - x$.

Now, $x^3 - x = \underset{2}{\varphi}$ wherefore,

3. $x^3 - \varphi - x = \underset{2}{\varphi} - \varphi = \varphi x$, and $\varphi x + \varphi = \underset{2}{\varphi}$.

4. $\varphi x^2 = (x^2 - x) \times x^2 = x^4 - x^3$,

And as $\varphi x^2 = \varphi\,(\varphi + x) = \varphi^2 + \varphi\, x$,

we have $x^4 - x^3 = \varphi^2 + \varphi\, x$.

5. But (3) $\varphi\, x = \underset{2}{\varphi} - \varphi$, consequently.

$$x^4 - x^3 = \varphi^2 + \underset{2}{\varphi} - \varphi.$$

6. Farther $x^4 = \underset{3}{\varphi} + x$, and $x^3 = \underset{2}{\varphi} + x$, so

$x^4 - x^3 = \underset{3}{\varphi} + x - \underset{2}{\varphi} - x = \underset{3}{\varphi} - \underset{2}{\varphi}$, and

we have therefore, (5) $\varphi^2 + \underset{2}{\varphi} - \varphi = \underset{3}{\varphi} - \underset{2}{\varphi}$

whence $\varphi^2 + 2\,\underset{2}{\varphi} - \varphi = \underset{3}{\varphi}$,

which we may prove thus,

$$\begin{array}{lll} \varphi^2 = (x^2 - x)^2 & = & x^4 - 2x^3 + x^2 \\ 2\,\underset{2}{\varphi} = x^3 - x \times 2 & = & 2x^3 - 2x \\ -\varphi & = & -x^2 + x \\ \hline \text{Sum} = \underset{3}{\varphi} & = & x^4 \;\; * \;\; * - x \end{array}$$

7. $\varphi x^3 = x^3\,(x^2 - x) = x^5 - x^4 = \mathrm{F} + x - \underset{3}{\varphi} - x = \mathrm{F} - \underset{3}{\varphi}$.

8. But as $\varphi x^3 = \varphi\,(\underset{2}{\varphi} + x) = \underset{2}{\varphi}\,\varphi + \varphi\, x$

and as (3) $\varphi x = \underset{2}{\varphi} - \varphi$ we have

$\underset{2}{\varphi}\,\varphi + \underset{2}{\varphi} - \varphi = \mathrm{F} - \underset{3}{\varphi}$, and

$\underset{2}{\varphi}\,\varphi + \underset{3}{\varphi} + \underset{2}{\varphi} - \varphi = \mathrm{F}$

Thus if $F=(3^5-3)=243-3=240$

$$\underset{2}{\varphi}\varphi=6\left(\frac{Tx-1}{4}\right)\times 2\left(\frac{Tx-1}{3}\right)=(6\times 4)\times 6=6\left(\frac{Tx-1}{4}\right)=$$

$24\times 6=$ . . .	144
$\underset{3}{\varphi}=81-3=$	78
$\underset{2}{\varphi}=6\times 4=$	24
	246
$-\varphi=-3\times 2=$	−6
	240

14. We may remark generally of the functions φ, that

$$\underset{2}{\varphi}=\varphi x+\varphi$$

$$\underset{5}{\varphi}=\underset{2}{\varphi}x+\varphi$$

$$F=\underset{4}{\varphi}=\underset{5}{\varphi}+\varphi$$

which we prove thus,

$\underset{2}{\varphi}=x^3-x$ and $\varphi x=x^3-x^2$

adding $\varphi = x^2-x$

we have $\varphi x+\varphi = x^3 * -x=\underset{2}{\varphi}$

$\underset{3}{\varphi}=x^4-x$ and $\underset{2}{\varphi}x=x^4-x^2$

adding $\varphi = x^2-x$

we have $\underset{2}{\varphi}x+\varphi=x^4 * -x=\underset{3}{\varphi}$

$\underset{4}{\varphi}=F=x^5-x$ and $\underset{3}{\varphi}x=x^5-x^2$

adding $\varphi = x^2-x$

we have $\underset{3}{\varphi}x+\varphi=x^5 * -x=\underset{4}{\varphi}=F$

15. We may multiply the functions φ into each other.

1. Thus $\underset{2}{\varphi}\,\varphi=(x^3-x)\times(x^2-x)=x^5-x^4-x^3+x^2$.

Thus if x be 4, φ will be 3.

$$\left.\begin{array}{l}\text{and } \underset{2}{\varphi}=10\times 6\\ \varphi=\ 6\times 2\end{array}\right\}=60\times 12=720$$

$=4^5=1024-x^4=-256-x^3=-64+x^2=16=1040-320=720$.

If $x=3$, $\varphi=2$, and we have

$$\left.\begin{array}{l}\underset{2}{\varphi}=4\times 6\\ \varphi=3\times 2\end{array}\right\}=24\times 6=144.$$

$=3^5=243-x^4=-81-x^3=-27+x^2=9=252-108=144$

16. $\underset{3}{\varphi}\times\varphi=(x^4-x)\,(x^2-x)=x^6-x^5-x^3+x^2$

Thus, $\left.\begin{array}{l}x=5\\ \varphi=4\end{array}\right\}$ here $\underset{3}{\varphi}=\underset{2}{\varphi}x+\varphi=6\binom{Tx-1}{4}\times x+2\binom{Tx-1}{3}$

$=(20\times 6)\times 5=600$

$+\ 10\times 2\ \ =\ 20$

Sum $\underset{3}{\varphi}=\ \ 620\times\varphi=(10\times 2)=12400$

$x^6=15625-x^5=-3125-x^3=-125+x^2=25=15650-3250$
$=12400-$

17. If we apply the figurate numbers, we shall find them equally serviceable,

$\varphi=(x^2-x)\times 2f=$ pentagonal numbers $=3\binom{Tx-1}{3}+x=$

$3x^2-x$, whence $(x^2-x)\,(3x^2-x)=3x^4-4x^3+x^2=\varphi\times 2f$.

Multiplying $\varphi=(x^2-x)$ by the hexagonal numbers $=2x^2-3x$, we obtain,

$$2x^4-3x^3+x^2=\underset{2}{\varphi}f.$$

Or multiplying by the heptagonal numbers, where $2f_3 = 5x^2 - 3x$, by $\varphi = (x^2 - x)$, we have $2\varphi f_3 = 5x^4 - 8x^3 + 3x^2$; and as the figurate numbers do not differ in degree, but only in the coefficients of the terms, we may always choose a figurate number having the same coefficient with the coefficient of the highest term in the equation to be resolved.

18. As there is no limit to the combinations which may be formed in this way, we may adapt them to any form or degree of an equation, and it is needless to proceed; for it is not my design, nor can I pretend at present to enter into this most extensive inquiry any farther, than to give a general outline of its nature and power, and the manner of its application to the solution of equations.

If, however, the methods of resolving equations in terms of the orders of numbers are sufficiently understood, the application of these substitutions will be found precisely similar, and equally easy; for all these functions may be found in terms of the orders of numbers.

I however subjoin one or two examples.

Required the value of x in this equation,

$$x^4 - 30x = 136.$$

here we may substitute $\varphi_3 + x = x^4$.

or $\varphi^2 = (x^2 - x)^2 = x^4 - 2x^3 + x^2$

or $\varphi x = (x^3 - x) \times x = x^4 - x^2$.

Taking this last substitution, we have, by subtracting from

$$x^4 - 30x - 136 = 0,$$
$$x^4 - x^2 - \tfrac{\varphi}{2}x = 0,$$

$$* \quad x^2 - 30x + \tfrac{\varphi}{2}x - 136 = 0 \text{ and substituting}$$

$\varphi + x = x^2$ " $x^2 - x - \varphi = 0$

We obtain $* \quad -29x + \tfrac{\varphi}{2}x + \varphi = 136$, and dividing by x

it becomes $\dfrac{-29 + \frac{\varphi}{2}x + \varphi}{x} = \dfrac{136}{x}$ and transposing 29

we have $\dfrac{\frac{\varphi}{2}x + \varphi}{x} = \dfrac{136}{x} + 29$, whence $\tfrac{\varphi}{2}x + \varphi$ is greater than 136.

$\tfrac{\varphi}{2}x = \binom{Tx-1}{4} = 0,\ 1,\ 4,\ 10,\ 20,\ 35,$ &c.

$\times 6x = \quad 6,\ 12,\ 18,\ 24,\ 30,\ 36,$ &c.

$0,\ 12,\ 72,\ 240,\ 600,$

adding $\varphi = 2\binom{Tx-1}{3} = 2,\ 6,\ 12,\ 20$

Series $\tfrac{\varphi}{2}x + \varphi = 14,\ 78,\ 252,\ 620$

divided by $x + = 2,\ 3,\ 4,\ 5$

$\dfrac{\frac{\varphi}{2}x + \varphi}{x} = 7,\ 26,\ 63,\ 124,$ &c. and taking $x = 4$,

we have $\dfrac{136}{4} + 29 = 34 + 29 = 63$, which therefore answers the conditions.

Required the value of x in this equation, $3x^4 - 5x^3 + 11x - 5565 = 0$.

Substituting $2f$ or twice the pentagonal numbers $=3x^2-x$, if we multiply this by $\varphi=(x^2-x)$, we have $3x^4-4x^3+x^2-2f\varphi=0$, which we shall subtract from the given equation,

$$=3x^4-5x^3+11x-5565=0$$
$$3x^4-4x^3+x^2-2f\varphi=0$$

and our equation becomes $=-x^3-x^2+11x+2f\varphi-5565=0$

Substituting $x^3=\frac{\varphi}{2}+x$ $\quad -x^3+x+\frac{\varphi}{2}=0$

we have $\quad -x^2+10x+2f\varphi-\frac{\varphi}{2}-5565=0$

Substituting $x^2=\varphi+x$ $\quad -x^2+x+\varphi=0$

$$9x+2f\varphi-\frac{\varphi}{2}-\varphi=5565$$

And transposing $\quad 2f\varphi-\frac{\varphi}{2}-\varphi=5565-9x$

Dividing by x $\quad \frac{2f\varphi-\frac{\varphi}{2}-\varphi}{x}=\frac{5565}{x}-9.$

So $2f\varphi-\frac{\varphi}{2}-\varphi$ is less than 5565.

$$2f=2\times 3\left(\frac{Tx-1}{3}\right)+x=6\left(\frac{Tx-1}{3}\right)+2x$$

Series $\left(\frac{Tx-1}{3}\right)$ = 0, 1, 3, 6, 10, 15, 21, 28, 36 &c.

$\times 6$ = 0, 6, 18, 36, 60, 90, 126, 168, 216 &c.

$+2x$ = 2, 4, 6, 8, 10, 12, 14, 16, 18 &c.

Series $2f$ = 2, 10, 24, 44, 70, 102, 140, 184, 334 &c.

$\times\varphi=2\left(\frac{Tx-1}{3}\right)$ = 0, 2, 6, 12, 20, 30, 42, 56, 72 &c.

Products $2f\varphi$ = 20, 144, 168, 1400, 3000, 5880, &c.; we may stop here, as 5880 exceeds 5565,

so $2f\varphi$ cannot exceed 5580, and $x=7$. } We take therefore $2f\varphi = 5880$

and subtract $\varphi+\underset{2}{\varphi}=$

$$6\binom{Tx-1}{4}+2\binom{Tx-1}{3}\Big\}=336$$

$=6$th term of 4th order $\times 6$

$+$ 6th term of 3d order $\times 2$ $\Big\}= 42$

$=21\times 2$

$= 378$

Remains $2f\underset{2}{\varphi}-\underset{2}{\varphi}-\varphi = 5502$

Dividing by $x=7$, we have

$$\frac{5502}{7}=786=\frac{5565}{7}-9=795-9=786.$$

Let there be given the equation

$$x^5-45x^3+40x^2+444x-720=0$$

$\underset{2}{\varphi}\varphi=x^5-x^4-x^3+x^2$. So we subtract

$$x^5-x^4-x^3+x^2-\underset{2}{\varphi}\varphi=0$$

$$x^4-44x^3+39x^2+444x+\underset{2}{\varphi}\varphi-720=0$$

$\varphi^2=x^4-2x^3+x^2$

$$x^4-2x^3+x^2-\varphi^2=0$$

$$-42x^3+38x^2+444x+\underset{2}{\varphi}\varphi+\varphi^2-720=0$$

$42\underset{2}{\varphi}=-42x^3+42x$

$$-42x^3+42x+42\underset{2}{\varphi}=0$$

$$38x^2+402x+\underset{2}{\varphi}\varphi+\varphi^2-42\underset{2}{\varphi}-720=0$$

$38\varphi=38x^2-38x$

$$38x^2-38x-38\varphi=0$$

$$440x+\underset{2}{\varphi}\varphi+\varphi^2+38\varphi-42\underset{2}{\varphi}-720=0$$

Whence we have $440x+\underset{2}{\varphi}\varphi+\varphi^2+38\varphi-42\underset{2}{\varphi}=720$

So we observe $\underset{2}{\varphi}\varphi+\varphi^2+38\varphi-42\underset{2}{\varphi}$, must be negative, unless x is unity or negative, for $440\times 2=880>720$.

We may abridge the calculation, and also learn the limits of the equation, by subtracting φ from 42, as φ is a factor of $\frac{\varphi}{2}$ in the affirmative part of the equation, and 42 is a factor of $\frac{\varphi}{2}$ in the negative part; and as 42φ must be in excess, unless x be negative, φ must be less than 42.

	Terms 1,	2,	3,	4,	5,	6,	7
Series of 42	=42,	42,	42,	42,	42,	42,	42
Series of $\varphi = 2\binom{Tx-1}{3}$	= 0,	2,	6,	12,	20,	30,	42
1. $42-\varphi$	=42,	40,	36,	30,	22,	12,	0

So x, if affirmative, cannot exceed 6.

Series $2\binom{Tx-1}{3}$	= 0,	2,	6,	12,	20,	30,	42
Series $\varphi^2 = 2\binom{Tx-1}{3}^2$	= 0,	4,	36,	144,	400,	900,	1764
Series $38\varphi = 76\binom{Tx-1}{3}$	= 0,	76,	228,	456,	760,	1140,	1596
$+440x =$	=440,	880,	1320,	1760,	2200,	2640,	3080
2. Sums $\varphi^2+38\varphi+440x$	=440,	960,	1584,	2360,	3360,	4680,	6440
Series $42-\varphi$ (1)	= 42,	40,	36,	30,	22,	12,	0
		×	×	×	×	×	
$\times\frac{\varphi}{2} = 6\binom{Tx-1}{4}$	= 0,	6,	24,	60,	120,	210,	&c.
3. Products $= (42-\varphi)\frac{\varphi}{2}$	= 0,	240,	864,	1800,	2640,	2520,	
Differences of 2d and 3d $= \varphi^2 + 38\varphi + 440x - (42-\varphi)\varphi^2$	=440,	720,	720,	560,	720,	2160,	&c.

We observe, therefore, that the three positive roots are 2, 3, 5; from which the other two roots, which are negative, viz. —4 and —6 may be found.

It is very evident, that there is a variety of substitutions of the same kind, which may be made at pleasure, and that all these expressions φ_1, φ_2, φ_3, F, F_s, &c. are convertible terms.

23. I shall give one or two examples of the application of this method to the irreducible case of Cardan's rule, where one of the roots is an integer.

1st.

Given	x^3-6x^2+10x	$=8$
Subtracting x^3, and adding its equivalent φ_2+x (φ_2-x)	$=x^3 \quad +x+\varphi_2$	$=0$
	$*\ -6x^2+11x+\varphi_2$	$=8$
Substituting $6\varphi=6x^2-6x$	$-6x^2+6x+6\varphi$	$=0$
	$*\quad *\quad 5x+\varphi_2-6\varphi$	$=8$
Whence transposing	$-6\varphi+\varphi_2-8$	$=-5x$
Changing the signs	$6\varphi-\varphi_2+8$	$=5x$
Dividing by 5	$\frac{6\varphi-\varphi_2+8}{5}$	$=x$

Wherefore $6\varphi-\varphi_2+8$ must be positive, and divisible by 5, if x is an integer and positive.

Terms 1, 2, 3, 4, 5, 6, 7, 8,

$6\varphi = 6 \times 2\binom{Tx-1}{3} = \binom{Tx-1}{3} = 0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,$ &c.

$\times 12 = 0, 12, 36, 72, 120, 180, 252, 336,$ &c.

$+\ 8 = 8,\ 8,\ 8,\ 8,\ 8,\ 8,\ 8,\ 8$

$6\varphi + 8 = 8, 20, 44, 80, 128, 188, 260, 344,$ &c.

$-\frac{\varphi}{2} = 6\binom{Tx-1}{4} = 0,\ 6, 24, 60, 120, 210, 336, 504,$ &c.

$6\varphi + 8 - \frac{\varphi}{2} = 8, 14, 20, 20,\ 8 - 22 - 64 - 160,$ &c.

So x can only be 3 or 4, since these terms $= 20$ are divisible by 5, and affirmative.

Trying 3, we have $x^3 - 6x^2 + 10x = 8$
$= 27 - 54 + 30 = 3$, which does not answer; but if $x = 4$ $= 64 - 96 + 40 = 8$.

2d.

Given $y^3 - 19y = 30$. Substituting $\frac{\varphi}{2} = y^3 - y$

$y^3 - y - \frac{\varphi}{2} = 30$

$-18y + \frac{\varphi}{2} = 30$ and transposing

we obtain $\frac{\varphi}{2} - 30 = 18y$ and dividing by 18

we have $\frac{\frac{\varphi}{2} - 30}{18} = y$, whence $\frac{\varphi}{2} - 30$ must be affirmative and divisible by 18, if y is an integer and affirmative.

Terms, 1, 2, 3, 4, 5, 6, 7,

$\frac{\varphi^2}{2} = 6\binom{Tx-1}{4} = 0,\ 6,\ 24,\ 60,\ 120,\ 180,\ 336,$ &c.

$-30 = -30, -30, -30, -30, -30, -30, -30,$ &c.

$-30, -24, -6, +30, +90, 150, +306,$ &c.

Dividing by 18, . . 5 . .

we observe that $y = 5$ succeeds; so we have

$y^3 - 19y = 125 - 95 = 30.$

24. The application of this method, therefore, to the resolution of equations, and especially to equations of the higher orders, is very simple, and similar to the examples given of the application of the theorems derived from the different orders of numbers. The advantage it possesses over these theorems is, its greater generality, and that it affords the means of resolving equations, in terms of any particular order of numbers, independently of the other orders of numbers. It also enables us to carry our equations to as high powers as we please, being attended with no other difficulty than what must ever arise from the application of ordinary arithmetic to very high powers; which, although they do not exceed the capabilities of its notation, are yet expressed by it very inconveniently; so that the calculations become, in some measure, impracticable by means of the natural numbers. Whether it might be practicable, or if practicable, whether it would be useful to extend for that purpose our arithmetic another step, by means of the 3d order of numbers, which might be denoted by the ordinary numbers accented or otherwise distinguished, I am too sensible of my own deficiencies in science to hazard any opinion.

I proceed to prove the perfect generality of this theorem, by shewing, that we may always find these orders of ϕ to be substituted for x in any equation. For this will be effected, if we can discover a general method of finding the orders of numbers.

VI.

A General Theorem for finding the xth term of any order of numbers m, *and which includes the case of the Binomial Theorem.*

1. If we multiply x by $\binom{x-1}{2}$ we obtain $\frac{x^2-x}{2} = \binom{\mathrm{T}x-1}{3}$, and if we multiply $\mathrm{T}\,{}_{3}^{x}$ by $\frac{x-1}{3}$ we obtain $\binom{\mathrm{T}x-1}{4}$.

For $\mathrm{T}\,{}_{3}^{x} = \frac{x^2+x}{2}$ and multiplying by $\frac{x-1}{3}$ we shall have this equation $\frac{x^3-x}{6} = \binom{\mathrm{T}x-1}{4}$; multiplying by 6 we have x^3-x $6\binom{\mathrm{T}x-1}{4}$ and $x^3 = 6\binom{\mathrm{T}x-1}{4} + x$.

In the same way $\mathrm{T}\,{}_{4}^{x} \times \frac{x-1}{4} = \binom{\mathrm{T}x-1}{5}$.

For $\mathrm{T}\,{}_{4}^{x} = \frac{(x+1)^3-(x+1)}{6} = \frac{x^3+3x^2+3x+1-(x+1)}{6} = \frac{x^3+3x^2+2x}{6}$, and multiplying by $\frac{x-1}{4}$, we obtain $\frac{x^4+2x^3-x^2-2x}{24}$. But the 2d formula for 4th powers (p. 48.), in terms of the orders of numbers gives

$x^4 = 12\binom{\mathrm{T}x-1}{5} + 12\binom{\mathrm{T}x-2}{5} + x^2$ wherefore we have

$$x^4 = 12\binom{\mathrm{T}x-1}{5} + 12\binom{\mathrm{T}x-2}{5} + x^2$$

$$\text{and } 2x^3 = 12\binom{\mathrm{T}x-1}{4} + 2x$$

$$-x^2 = -x^2$$

$$-2x = -2x$$

$$x^4+2x^3-x^2-2x = 12\binom{\mathrm{T}x-1}{5} + 12\binom{\mathrm{T}x-2}{5} + 12\binom{\mathrm{T}x-1}{4}$$

But $12\binom{Tx-2}{5}+12\binom{Tx-1}{4} = 12\binom{Tx-1}{5}$, therefore

$= 12\binom{Tx-1}{5}+12\binom{Tx-1}{5} = 24\binom{Tx-1}{5}$

and $\frac{Tx}{4}\times\frac{x-1}{4}=\frac{24}{24}\binom{Tx-1}{5} = \binom{Tx-1}{5}$.

2. We might demonstrate in the same way that $\frac{Tx}{5}\times\left(\frac{x-1}{5}\right)=\binom{Tx-1}{6}$ and that $\frac{Tx}{6}\times\left(\frac{x-1}{6}\right)=\binom{Tx-1}{7}$ &c. and generally $\frac{Tx}{m}\times\left(\frac{x-1}{m}\right)=\binom{Tx-1}{m+1}$.

3. To apply this rule,

$x\times\frac{x-1}{2}=\binom{Tx-1}{3}$. Let $x=2$ $\quad 2\times\frac{1}{2}=1$

$x=3$ $\quad 3\times\frac{2}{2}=3$

$x=4$ $\quad 4\times\frac{3}{2}=6$

$x=5$ $\quad 5\times\frac{4}{2}=10$, &c.; generally making $x=m$ we have $\frac{m\,.\,m-1}{1\quad 2}=\binom{Tx-1}{3}$.

$\frac{Tx}{6}\times\left(\frac{x-1}{2}\right)=\binom{Tx-1}{4}$. Thus,

If $x=2$ $\frac{Tx}{3}= 3=\frac{3\,.\,2}{1\,.\,2}\times\frac{2-1}{3}=\frac{3\,.\,2\,.\,1}{1\,.\,2\,.\,3}= 1=\binom{Tx-1}{4}$

$x=3$ $\frac{Tx}{3}= 6=\frac{4\,.\,3}{1\,.\,2}\times\frac{3-1}{3}=\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}= 4$

$x=4$ $\frac{Tx}{3}=10=\frac{5\,.\,4}{1\,.\,2}\times\frac{4-1}{3}=\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}=10$

$x=5$ $\frac{Tx}{3}=15=\frac{6\,.\,5}{1\,.\,2}\times\frac{5-1}{3}=\frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}=20$, &c.

making $x+1=m$, we have $\frac{m\,.\,m-1\,.\,m-2}{1\;.\;2\;.\;3}=\binom{T\,.\,m-2}{4}$

3d. $\overset{\mathrm{T}x}{4}\times\left(\frac{x-1}{4}\right)=\left(\overset{\mathrm{T}x-1}{5}\right)$. Thus,

if $x=2$ $\overset{\mathrm{T}x}{4}=\frac{4.3.2}{1.2.3}\times\frac{x-1}{4}=\frac{4.3.2.1}{1.2.3.4}=1=\left(\overset{\mathrm{T}x-1}{5}\right)$

$x=3$ $\overset{\mathrm{T}x}{4}=\frac{5.4.3}{1.2.3}\times\frac{3-1}{4}=\frac{5.4.3.2}{1.2.3.4}=5$

$x=4$ $\overset{\mathrm{T}x}{4}=\frac{6.5.4}{1.2.3}\times\frac{4-1}{4}=\frac{6.5.4.3}{1.2.3.4}=15,$ &c.

and making $x+2=m$, we have $\frac{m.m-1.m-2.m-3}{1.2.3.4}=\left(\overset{\mathrm{T}m-3}{5}\right)$

2. In the same way we shall find, making $x+3=m$,

that $\frac{m.m-1.m-2.m-3.m-4}{1.2.3.4.5}=\left(\overset{\mathrm{T}m-4}{6}\right)$, &c.

and the law being evident, we may construct the following table.

For 2d order of numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. $=\frac{x}{1}$

3. Order, 0, 1, 2, 3, 4, 5, 6, 7, 8, $+2=\frac{x+1\times x.}{1,\ 2}$

4. Order, 0, 0, 1, 2, 3, 4, 5, 6, 7, $+3=\frac{x+2\times x+1\times x}{1,\ 2,\ 3}$

5. Order, 0, 0, 0, 1, 2, 3, 4, 5, 6, $+4=\frac{x+3\times x+2\times x+1\times x}{1\ \ 2,\ \ 3,\ \ 4}$

6. Order, 0, 0, 0, 0, 1, 2, 3, 4, 5, $+5=\frac{x+4\times x+3\times\ .\ .\ x}{1,\ 2,\ .\ .\ .\ 5}$

7. Order, 0, 0, 0, 0, 0, 1, 2, 3, 4, $+6=\frac{x+5\times x+4\times\ .\ .\ x}{1,\ 2,\ .\ .\ .\ 6}$

8. Order, 0, 0, 0, 0, 0, 0, 1, 2, 3, $+7=\frac{x+6\times x+5\times\ .\ .\ x}{1,\ 2,\ .\ .\ .\ 7}$

9. Order, 0, 0, 0, 0, 0, 0, 0, 1, 2, $+8=\frac{x+7\times x+6\times\ .\ x}{1,\ 2,\ .\ .\ 8}$

10. Order, 0, 0 0, 0, 0, 0, 0, 0, 1, $+9=\frac{x+8\times x+7\times\ .\ .\ x}{1\ \ 2\ .\ .\ .\ 9}$

In this table, x denotes the lowest number in each column. Thus, in the 3d order, or 2d horizontal line, if $x=1$, it will be found in the 2d vertical column, having $x+1=2=m$, at the top of the column, if $x=4$, and $x+1=5=m$, is at the top of the column.

In the 4th order, or 3d horizontal line, $x+2=m$ is at the top of the column. Thus if $x=1$, 3 is at the top of the column; if $x=3$, $m=x+2=5$, is at the top of the column, and so forth.

We are thus enabled to find the term of any order of numbers by this table.

If it were required, for example, to find the term x of the order m, we are to increase x by $m-2$, for the top figure of the column, and take $m-1$ factors decreasing by a series of units from $x+m-2$, and divide the product of these factors by the product of $m-1$ terms of the series 1, 2, 3, 4, &c.

Thus let there be required the xth term of the 4th order, here $m=4$, and $(x+m-2)=x+2$; so we have $\frac{x+2\times x+1\times x}{1\,.\,2\,.\,3}=\mathrm{T}_4^x$; or calling the figure at the top of the column m we have $\frac{m\,.\,m-1\,.\,m-2}{1\,.\,2\,.\,3}=\left(\mathrm{T}_4^{m-2}\right)$.

Generally, we may express our general formula in terms of x, thus,

$$\frac{x+m-2\times x+m-3\times x+m-4, \text{ \&c. for } m-1 \text{ terms.}}{1\,.\quad 2\,.\quad 3\,.\quad m-1}$$

3. This theorem affords a very easy demonstration of the binomial theorem, which is only a particular case where the number of the term decreases by an arithmetical progression of units; while the order of the term increases by the same arithmetical progression; and the terms and orders of numbers being reciprocals, the highest factor of all the terms of a binomial remains

constant; while the number of the factors and of the divisors increase, until the whole number of the terms are exhausted; and the number of terms in a binomial power is always expressed by the index of the power $m+1$.

4. This may, however, require some farther explanation. We have already remarked, in the preliminary chapter, when treating of the orders of numbers and the binomial table, that the coefficients of the 1st terms of the successive powers of a binomial form the series of units, or 1st order of numbers,—That the coefficients of the series of 2d terms of the successive powers of a binomial form the series of natural numbers, or numbers of the 2d order; that the series of the coefficients of the binomial third terms form the series of trigonal numbers, or numbers of the 3d order; the series of coefficients of 4th terms, the 4th order of numbers, and so on; so that the series of binomial coefficients, and of the orders of numbers, are identical and synonymous.

5. We have also noticed, in the same chapter, that the terms of the different orders of numbers, and the orders of the terms, are reciprocals. Thus the 1st term of the 9th order = 9th term 1st order; 3d term 2d order = 2d term 3d order; 7th term 5th order = 5th term 7th order; and generally the nth term of the mth order = mth term of the nth order.

6. We have also seen, that the terms of any number of units x, raised to the power m, is always equal to $mx-(m-1)$, (*vide* p. 27.); consequently, as $x=2$ in a binomial power, we have, for the number of terms of a binomial raised to the power m, $2m-(m-1)=m+1$.

We have also remarked, on the binomial table, that the series of binomial terms are the diagonal terms of the table, reading

from the 1st term of the $(m+1)$ order upon the left upwards, to the $(m+1)$st term of the 1st order at the top upon the right; so that the orders of the terms decrease as the numbers of the terms increase.

8. We may therefore demonstrate the binomial theorem, by shewing that it is included in the theorem

$$\frac{x+m-2+x+m-3, \text{ \&c. for } m-1 \text{ terms}}{1 \cdot 2 \cdot 3 \ldots m-1.}$$

Let it be required to find by this theorem the coefficients of the terms of a binomial raised to the 6th power, where $m=6$.

1st, The number of the terms are $6+1=7=\binom{\mathrm{T}m+1}{1}$.

The 1st term = 7th term of the 1st order = $1=\frac{m+1}{m+1}$

the 2d term = 6th term of the 2d order = $6=\frac{m}{1}$ = $\binom{\mathrm{T}m}{2}$

the 3d term = 5th term of the 3d order = $15=\frac{m \cdot m-1}{1 \cdot 2}$ = $\binom{\mathrm{T}m-1}{3}$

the 4th term = 4th term of the 4th order = $20=\frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3}$ = $\binom{\mathrm{T}.m-2}{4}$

the 5th term = 3d term of the 5th order = $15=\frac{m \cdot m-1 \cdot m-2 \cdot m-3}{1 \cdot 2 \cdot 3 \cdot 4}$ = $\binom{\mathrm{T}.m-3}{5}$

the 6th term = 2d term of the 6th order = $6=\frac{m \cdot m-1 \ldots m-4}{1.2 \ldots 5}$ = $\binom{\mathrm{T}m-4}{6}$

the 7th term = 1st term of the 7th order = $1=\frac{m \cdot m \ldots . m-5}{1.2 \ldots . 6}$ = $\binom{\mathrm{T}m-5}{7}$

9. It is scarcely necessary to shew, for it is an identical proposition, that if $x=2$, $x+m-2=m$, and x is always 2 in the case of the powers of a binomial. Nor, consequently, that the theorem $\frac{x+m-2}{1} \times \frac{x+m-3}{2}$, &c. &c. includes in it the binomial the-

orem, where the index of the power is a fraction as $\frac{m}{n}$; for still we have $x=2$, and the theorem gives $2+\frac{m}{n}-2=\frac{m}{n}$ for the highest factor, and the next factor is, by the general rule, $\frac{\frac{m}{n}-1}{2}$, the 3d factor, $\frac{\frac{m}{n}-2}{3}$, &c. as in the binomial theorem.

10. We may also demonstrate the binomial theorem in a very simple and beautiful manner, by means of the following table of the orders of numbers, found by the continual multiplication of a series of fractional terms:

1st	Order $=1, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \frac{9}{9}, \frac{10}{10}$, &c. $=$	1, 1, 1, 1, &c.
2d	Order $=1, \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}$, &c. $=$	1, 2, 3, 4, &c.
3d	Order $=1, \frac{3}{1}, \frac{4}{2}, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \frac{8}{6}, \frac{9}{7}, \frac{10}{8}$, &c. $=$	1, 3, 6, 10, &c.
4th	Order $=1, \frac{4}{1}, \frac{5}{2}, \frac{6}{3}, \frac{7}{4}, \frac{8}{5}, \frac{9}{6}, \frac{10}{7}$, &c. $=$	1, 4, 10, 20, &c.
5th	Order $=1, \frac{5}{1}, \frac{6}{2}, \frac{7}{3}, \frac{8}{4}, \frac{9}{5}, \frac{10}{6}$, &c. $=$	1, 5, 15, 35, &c.
6th	Order $=1, \frac{6}{1}, \frac{7}{2}, \frac{8}{3}, \frac{9}{4}, \frac{10}{5}$, &c. $=$	1. 6, 21, 56, &c.
7th	Order $=1, \frac{7}{1}, \frac{8}{2}, \frac{9}{3}, \frac{10}{4}$, &c. $=$	1, 7, 28, 84, &c.
8th	Order $=1, \frac{8}{1}, \frac{9}{2}, \frac{10}{3}$, &c. $=$	1, 8, 36, 120, &c.
9th	Order $=1, \frac{9}{1}, \frac{10}{2}$, &c. $=$	1, 9, 45, 165, &c,
10th	Order $=1, \frac{10}{1}$, &c. $=$	1, 10, 55, 220, &c.
11th	Order $=1$, &c. $=$	1, 11, 66, 286, &c.
	&c.	

11. We may observe, from this table, not only the identity of the binomial theorem with the terms of the table, but the *rationale* of the theorem itself.

Beginning with the uppermost line of the table, or the line of the 1st order of numbers, we observe, that the term is always the number divided by itself, $= \frac{m}{m}$ or unity.

12. Descending to the 2d line, or the line of the 2d order, we observe, that the term is always the number divided by unity. This will be evident on the slightest consideration of the series of terms; but an example will at once make it evident. Let $m=5$; then looking in the 2d line in the diagonal line of 5, we find $\frac{5}{4}$; and the whole preceding factors of that line are $\frac{5}{4}\times\frac{4}{3}\times\frac{3}{2}+\frac{2}{1}\times\frac{1}{1}$, where we observe, that all the factors of the numerator are also in the denominator, except 5, the number itself, and that the only factor of the denominator that is not also in the numerator, is unity; so that the whole factors of the term, when reduced, $= \frac{5}{1} = \frac{m}{1}$.

13. When we come to the 3d line, or series of the 3d order, we observe, that there are two factors of the numerators which are not among the factors of the denominator, and two factors of the denominator which are not among the factors of the numerator, viz. the two highest factors of the numerator, and the two lowest of the denominator. Thus, let m be 5, the term of the 3d order is $\frac{5}{3}\times\frac{4}{2}\times\frac{3}{1}$, where 5 and 4 are not in the denominator, nor 1. and 2. in the numerator, the term, when reduced, is therefore $\frac{5\times 4}{1\,.\,2} = \frac{m\times m-1}{1\,.\,2\,.}$.

14. In the same way, at the 4th line, or 4th order of numbers, there are three factors in the numerator which are not in the denominator of the term, and three factors in the denominator which are not in the numerator of the term, viz. the three high-

est factors of the numerator and three lowest of the denominator; so the term, when reduced, is $\frac{m \times m-1 \times m-2}{1 \cdot 2 \cdot 3 \cdot}$, &c.

15. But besides this exposition of the law of the binomial theorem, we learn, from this table, to find the series of any order of numbers whatever with facility, the formula being the counterpart of the binomial theorem.

$1, \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{m+3}{4} \times \frac{m+4}{5}$, &c.; which series is equal to this other series

$$= 1, + \frac{m}{1} + \frac{m \cdot m+1}{1 \cdot 2} + \frac{m \cdot m+1 \cdot m+2}{1 \cdot 2 \cdot 3} + \frac{m \cdot m+1 \cdot m+2 \cdot m+3}{1 \cdot 2 \cdot 3 \cdot 4}.$$

16. Farther, we learn from it to find any term of any order of numbers, viz. the xth term of the mth order, for the numbers of terms are $m-1$, and the factors of the term are $\frac{x+m-2, \times x+m-3, \times x+m-4,}{1 \quad . \quad 2 \quad . \quad 3}$ &c. Thus, required 7th term of the 4th order. Here there are $4-1$ equal 3 terms; and these are $\frac{7+2, \times 7+1, \times 7}{1 \cdot 2 \cdot 3} = \frac{9 \times 8 \times 7}{1.2.3} = 3 \times 4 \times 7 = 84$, and consequently the sum of 7 terms of the 4th order is obtained by taking the 7th term of the 5th order. Here $m-1 = 4$ terms.

So the 7th term is $7+5-2 = \frac{10 \times 9 \times 8 \times 7}{1 \cdot 2 \cdot 3 \cdot 4} = 10.3.7 = 210 = 84 + 56 + 35 + 20 + 10 + 4 + 1 =$.

17. This method of finding the orders of numbers by the continual multiplication of a series of the natural numbers, in arith-

metical progression, has led me to inquire whether it could not be extended farther. I have had no leisure, however, to pursue the inquiry, and can only state the results of a very partial and limited investigation.

18. The series of odd numbers multiplied by the preceding terms of the series of natural numbers, gives the series of trigonals of the even numbers.

Series of odd numbers,	1, 3, 5, 7, 9, 11, 13, &c.
Series of natural numbers $x-1$	0, 1, 2, 3, 4, 5, 6,
Series of trigonals, being those of the even numbers,	0, 3, 10, 21, 36, 55, 78.

19. The series of odd numbers $\times$ by the series of natural numbers gives the series of the trigonals of the odd numbers.

Series of odd numbers =	1, 3, 5, 7, 9, 11, 13, &c.
Series of natural numbers =	1, 2, 3, 4, 5, 6, 7
Series of trigonals of odd numbers,	1, 6, 15, 28, 45, 66, 91.

20. Series of trigonal numbers multiplied by the preceding terms of the same series, $= \frac{x^4-x}{4}$, and divided by the product of the first two trigonal numbers, gives the series of factors of 4th powers. (*Vide* p. 47.)

Series of trigs. $= \frac{Tx}{3}$	1, 3, 6, 10, 15, 21, 28, &c. $4\left(\frac{Tx \times Tx-1}{3}\right)$
Ser. of trigs. $= \frac{Tx-1}{3}$	0, 1, 3, 6, 10, 15, 21 &c. $+2\left(\frac{Tx-1}{3}\right)+x$
Products,	0, 3, 18, 60, 150, 315, 588 &c. $= \frac{x^4-x}{4}$
Divided by 3,	0, 1, 6, 20, 50, 105, 196 &c.

which we know to be equal to each of the four following formulæ (vide p. 47.)

$$=\binom{Tx}{5}+\binom{Tx-1}{5}=2\binom{Tx-1}{5}+\binom{Tx}{4}=2\binom{Tx}{5}-\binom{Tx}{4}=$$
$$\binom{Tx}{6}-\binom{Tx-2}{6}.$$

21. But we may likewise compare this series with the series of the trigonals themselves, or 3d order of numbers.

3d Order =	1, 3, 6, 10, 15, 21, 28, 36
Trigonals of 3d order =	1, 6, 21, 55, 120, 231, 406, 666, &c.
Subtract the series	1, 6, 20, 50, 105, 196, 336, 540, &c.
Differences = 5th order of numbers	0, 0, 1, 5, 15, 35, 70, 126, &c.

So we have the series of products of two contiguous trigonal numbers divided by 3, equal to the trigonals of the trigonal numbers diminished by the series $Tx-2$ of the 5th order, or $\frac{Tx\times\binom{Tx-1}{3}}{3}=T\binom{Tx}{3}-\binom{Tx-2}{5}$, and by comparing it with the formula $\binom{Tx}{5}+\binom{Tx-1}{5}=\frac{\binom{Tx}{3}\times\binom{Tx-1}{3}}{3}$, we have

$T.\binom{Tx}{3}-\binom{Tx-2}{5}=\binom{Tx}{5}+\binom{Tx-1}{5}$, whence

$$T.\binom{Tx}{3}=\binom{Tx}{5}+\binom{Tx-1}{5}+\binom{Tx-2}{5}=\binom{Tx}{6}-\binom{Tx-3}{6}.$$

Thus let $x=7$ $T.T.7=T.28=406=\binom{T7}{5}+\binom{T6}{5}+\binom{T5}{5}=$
$=210+126+70=406$
$=\binom{Tx}{6}-\binom{Tx-3}{6}=\binom{T7}{6}-\binom{T4}{6}=462-56=406.$

22. We may also multiply the three contiguous terms of the trigonal numbers into each other, and divide by the product of the three first trigonals 1, 3, 6, thus:

Series 1, 3, 6, 10, 15, 21, 28, 36+1
0, 1, 3, 6, 10, 15, 21, 28+1 . 3
0, 0, 1, 3, 6, 10, 15, 21+1 . 3 . 6.

The terms will be $\frac{6.3.1}{1.3.6}=1$, $\frac{10.6.3}{1.3.6}=10$, $\frac{15.10.6}{1.3.6}=$

$5.10=50$.

$\frac{21.15.10}{1.3.6}=7.\frac{150}{6}=7.25=175$, $\frac{28.21.15}{1.3.6}=\frac{28.15.7}{2.3.1}=$
$14.5.7=70.7=490$, &c.
So the series is 1, 10, 50, 175, 490, &c.
which we shall more easily find by the multiplication of two contiguous terms of the 4th order, divided by the product of the two first terms $1.4=4$. Thus,

Series $\overset{Tx}{4}$ $=1, 4, 10, 20, 35, 56, 84, 120$, &c.

Series $\overset{Tx-1}{4}$ $=0, 1, 4, 10, 20, 35, 56, 84,$

Products $=0, 4, 40, 200, 700, 1960, 4704, 10080$

Terms $+4=$ series as before $\}=0, 1, 10, 50, 175, 490, 1176, 2520$, &c.

23. If we try to analyse this series by taking its successive differences, we shall not succeed; but

If we take the trigonals of the 4th order of numbers, thus,

Series $\mathrm{T}\,{}^{x}_{4}$ = 1, 4, 10, 20, 35, 56, 84, 120, &c.

Series $\mathrm{T}\,{}^{}_{3}$, of $\mathrm{T}\,{}^{x}_{4}$ = 1, 10, 55, 210, 630, 1596, 3570, 7260, &c.

and subtracting the above series } 1, 10, 50, 175, 490, 1176, 2520, 4950, &c.

we obtain the series = 0, 0, 5, 35, 140, 420, 1050, 2310, &c.

And dividing by 5 we obtain $\mathrm{T}\,{}^{x-2}_{7}$ } 0, 0, 1, 7, 28, 84, 210, 462, &c.

So we have $\left(\frac{\mathrm{T}\,{}^{x}_{3} \times \mathrm{T}\,{}^{x-1}_{3} \times \mathrm{T}\,{}^{x-2}_{3}}{18}\right) = \left(\frac{\mathrm{T}\,{}^{x}_{4} \times \mathrm{T}\,{}^{x-1}_{4}}{4}\right) =$

$\left(\mathrm{T}\,{}^{}_{3}\ \mathrm{T}\,{}^{x}_{4}\right) - 5\left(\mathrm{T}\,{}^{x-2}_{7}\right)$.

24. This same series may also be found by subtracting the 4th order of the trigonal numbers from the trigonals of the 4th order.

Thus trigonals = 1, 3, 6, 10, 15, 21, 28, &c.

$\left(\mathrm{T}\,{}^{}_{4}\ \mathrm{T}\,{}^{x}_{3}\right)$ = 1, 10, 56, 220, 680, 1771, 4060, &c.

Subtract $\left(\mathrm{T}\,{}^{}_{3}\ \mathrm{T}\,{}^{x}_{4}\right)$ = 1, 10, 55, 210, 630, 1596, 3570, &c.

Remains series found above, } = 0, 0, 1, 10, 50, 175, 490, &c.

There appears, indeed, to be an almost endless variety of these relations among the orders of numbers, many of which are, no doubt, merely curious; but some of them may also lead to important improvements in algebra, in the hands of those who, by their skill and knowledge, are enabled to derive the most advantage from the use of any new discovery.

If my labours should be attended with this success, I shall be sufficiently rewarded, in having contributed any thing towards their attainments; and here, I must for the present conclude this memoir, as my professional duties will not admit of my entering now upon the other subjects proposed to be treated of in the outline of this paper.

It may be some apology to state, that when I began this Memoir, I intended only to give a syllabus of the theorems for resolving equations in terms of the orders of numbers, derived from the synthesis and analysis of the powers of numbers; together with some account of the method of finding the prime factors of numbers, and of the tables for division and multiplication, with the theorem for the quadrature of the circle; and that I have been almost imperceptibly diverted from this design in the course of writing out this memoir for the press, by the circumstance of the miscellaneous theorems having first occurred to me, and afterwards the more general method of resolving equations in terms of the order of φ, and the theorems of finding the orders of numbers themselves, by an extension of the binomial theorem.

This enlargement of my original design, together with the necessity I was under of having this memoir printed off almost sheet by sheet, from the want of a sufficient number of types necessary for the publication of a mathematical work, will, I hope, entitle me to some indulgence for the want of more accuracy and clearness in the execution and arrangement of

this memoir. But there is one mistake into which I have fallen, which it is necessary that I should explain more particularly.

In the quotation given at p. 9 and 10, from Sir Isaac Newton's letter to Mr Oldenburgh, and in the observation made upon it, that Sir Isaac might not have observed the summations of the binomial coefficients to be the successive powers of $1+1$, as well as that they formed (according to the decimal notation of the figures) the powers of 11; I find I have been misled by taking the passage from Mr Hutton's Mathematical Dictionary, where it is quoted; for I have looked since into the original letter in Wallisii Opera, Vol. III. p. 634.; and I observe that Sir Isaac Newton has guarded most pointedly against the possibility of this misinterpretation. His words are, "Adeoque solæ numeratorum coefficientes numerales essent "investigandæ. Hæ autem in alternis datis areis erant figuræ "potestatum numeri 11, (*intelligo* $1+1$, *et sic in sequentibus*);" —the words included within the parenthesis have been omitted in Mr Hutton's translation of the passage.

Edinburgh, 13 *Elder Street*, *October* 1814.

TABLES OF THE ORDERS OF NUMBERS.

TABLE I.

TABLE OF THE THIRD, FOURTH, AND FIFTH ORDERS OF NUMBERS, FROM 1 TO 200.

Terms.	3d Order.	4th Order.	5th Order.	Terms.	3d Order.	4th Order.	5th Order.
1	1	1	1	23	276	2300	14,950
2	3	4	5	24	300	2600	17,550
3	6	10	15	25	325	2925	20,475
4	10	20	35	26	351	3276	23,751
5	15	35	70	27	378	3654	27,405
6	21	56	126	28	406	4060	31,465
7	28	84	210	29	435	4495	35,960
8	36	120	330	30	465	4960	40,920
9	45	165	495	31	496	5456	46,376
10	55	220	715	32	528	5984	52,360
11	66	286	1,001	33	561	6545	58,905
12	78	364	1,365	34	595	7140	66,045
13	91	455	1,820	35	630	7770	73,815
14	105	560	2,380	36	666	8436	82,251
15	120	680	3,060	37	703	9139	91.390
16	136	816	3,876	38	741	9880	101,270
17	153	969	4,845	39	780	10,660	111,930
18	171	1140	5,985	40	820	11,480	123,410
19	190	1330	7,315	41	861	12,341	135,751
20	210	1540	8,855	42	903	13,244	148,995
21	231	1771	10,626	43	946	14,190	163,185
22	253	2024	12,650	44	990	15,180	178,365

Terms.	3d Order.	4th Order.	5th Order.	Terms.	3d Order.	4th Order.	5th Order.
45	1035	16,215	194,580	84	3570	102,340	2,225,895
46	1081	17,296	211,876	85	3655	105,995	2,331,890
47	1128	18,424	230,300	86	3741	109,736	2,441,626
48	1176	19,600	249,900	87	3828	113,564	2,555,190
49	1225	20,825	270,725	88	3916	117,480	2,672,670
50	1275	22,100	292,825	89	4005	121,485	2,794,155
51	1326	23,426	316,251	90	4095	125,580	2,919,735
52	1378	24,804	341,055	91	4186	129,766	3,049,501
53	1431	26,235	367,290	92	4278	134,044	3,183,545
54	1485	27,720	395,010	93	4371	138,415	3,321,960
55	1540	29,260	424,270	94	4465	142,880	3,464,840
56	1596	30,856	455,126	95	4560	147,440	3,612,280
57	1653	32,509	487,635	96	4656	152,096	3,764,376
58	1711	34,220	521,855	97	4753	156,849	3,921,225
59	1770	35,990	557,845	98	4851	161,700	4,082,925
60	1830	37,820	595,665	99	4950	166,650	4,249,575
61	1891	39,711	635,376	100	5050	171,700	4,421,275
62	1953	41,664	677,040	101	5151	176,851	4,598,126
63	2016	43,680	720,720	102	5253	182,104	4,780,230
64	2080	45,760	766,480	103	5356	187,460	4,967,690
65	2145	47,905	814,385	104	5460	192,920	5,160,610
66	2211	50,116	864,501	105	5565	198,485	5,359,095
67	2278	52,394	916,895	106	5671	204,156	5,563,251
68	2346	54,740	971,635	107	5778	209,934	5,773,185
69	2415	57,155	1,028,790	108	5886	215,820	5,989,005
70	2485	59,640	1,088,430	109	5995	221,815	6,210,820
71	2556	62,196	1,150,626	110	6105	227,920	6,438,740
72	2628	64,824	1,215,450	111	6216	234,136	6,672,876
73	2701	67,525	1,282,975	112	6328	240,464	6,913,340
74	2775	70,300	1,353,275	113	6441	246,905	7,160,245
75	2850	73,150	1,426,425	114	6555	253,460	7,413,705
76	2926	76,076	1,502,501	115	6670	260,130	7,673,835
77	3003	79,079	1,581,580	116	6786	266,916	7,940,751
78	3081	82,160	1,663,740	117	6903	273,819	8,214,570
79	3160	85,320	1,749,060	118	7021	280,840	8,495,410
80	3240	88,560	1,837,620	119	7140	287,980	8,783,390
81	3321	91,881	1,929,501	120	7260	295,240	9,078,630
82	3403	95,284	2,024,785	121	7381	302,621	9,381,251
83	3486	98,770	2,123,555	122	7503	310,124	9,691,375

Terms.	3d Order.	4th Order.	5th Order.	Terms.	2d Order.	4th Order.	5th Order.
123	7626	317,750	10,009,125	162	13,203	721,764	29,772,765
124	7750	325,500	10,334,625	163	13,366	735,130	30,507,895
125	7875	333,375	10,668,000	164	13,530	748,660	31,256,555
126	8001	341,376	11,009,376	165	13,695	762,355	32,018,910
127	8128	349,504	11,358,880	166	13,861	776,216	32,795,126
128	8256	357,760	11,716,640	167	14,028	790,244	33,585,370
129	8385	366,145	12,082,785	168	14,196	804,440	34,389,810
130	8515	374,660	12,457,445	169	14,365	818,805	35,208,615
131	8646	383,306	12,840,751	170	14,535	833,340	36,041,955
132	8778	392,084	13,232,835	171	14,706	848,046	36,890,001
133	8911	400,995	13,633,830	172	14,878	862,924	37,752,925
134	9045	410,040	14,043,870	173	15,051	877,975	38,630,900
135	9180	419,220	14,463,090	174	15,225	893,200	39,524,100
136	9316	428,536	14,891,626	175	15,400	908,600	40,432,700
137	9453	437,989	15,329,615	176	15,576	924,176	41,356,876
138	9591	447,580	15,777,195	177	15,753	939,929	42,296,805
139	9730	457,310	16,234,505	178	15,931	955,860	43,252,665
140	9870	467,180	16,701,685	179	16,110	971,970	44,224,635
141	10,011	477,191	17,178,876	180	16,290	988,260	45,212,895
142	10,153	487,344	17,666,220	181	16,471	1,004,731	46,217,626
143	10,296	497,640	18,163,860	182	16,653	1,021,384	47,239,010
144	10,440	508,080	18,671,940	183	16,836	1,038,220	48,277,230
145	10,585	518,665	19,190,605	184	17,020	1,055,240	49,332,470
146	10,731	529,396	19,720,001	185	17,205	1,072,445	50,404,915
147	10,878	540,274	20,260,275	186	17,391	1,089,836	51,494,751
148	11,026	551,300	20,811,575	187	17,578	1,107,414	52,602,165
149	11,175	562,475	21,374,050	188	17,766	1,125,180	53,727,345
150	11,325	573,800	21,947,850	189	17,955	1,143,135	54,870,480
151	11,476	585,276	22,533,126	190	18,145	1,161,280	56,031,760
152	11,628	596,904	23,130,030	191	18,336	1,179,616	57,211,376
153	11,781	608,685	23,738,715	192	18,528	1,198,144	58,409,520
154	11,935	620,620	24,359,335	193	18,721	1,216,865	59,626,385
155	12,090	632,710	24,992,045	194	18,915	1,235,780	60,862 165
156	12,246	644,956	25,637,001	195	19,110	1,254,890	62,117,055
157	12,403	657,359	26,294 360	196	19,306	1,274,196	63,391,251
158	12,561	669,920	26,964,280	197	19,503	1,293,699	64,684,950
159	12,720	682,640	27,646,920	198	19,701	1,313,400	65,998,350
160	12,880	695,520	28,342,440	199	19,900	1,333,300	67,331,650
161	13,041	708,561	29,051,001	200	20,100	1,353,400	68,685,050

TABLE II.

TABLE OF THE SIXTH, SEVENTH, EIGHTH, NINTH, TENTH, ELEVENTH, AND TWELFTH ORDERS, FROM 1 TO 50.

Terms.	6th Order.	7th Order.	8th Order.	9th Order.	10th Order.	11th Order.	12th Order.
1	1	1	1	1	1	1	1
2	6	7	8	9	10	11	12
3	21	28	36	45	55	66	78
4	56	84	120	165	220	286	364
5	126	210	330	495	715	1001	1365
6	252	462	792	1287	2,002	3003	4368
7	462	924	1716	3,003	5,005	8008	12,376
8	792	1,716	3432	6,435	11,440	19,448	31,824
9	1,287	3,003	6435	12,870	24,310	43,758	75,582
10	2,002	5,005	11,440	24,310	48,620	92,378	167,960
11	3,003	8,008	19,448	43,758	92,378	184,756	352 716
12	4,368	12,376	31,824	75,582	167,960	352 716	705,432
13	6,188	18,564	50,388	125,970	293 930	646,646	1,352,078
14	8,568	27,132	77,520	203,490	497,420	1,144,066	2,496,144
15	11,628	38,760	116,280	319,770	817,190	1,961,256	4,457,400
16	15,504	54,264	170,544	490,314	1,307,504	3,268,760	7,726,160
17	20,349	74,615	245,157	735,471	2,042,975	5 311,735	13,037,895
18	26,334	100,947	346,104	1,081,575	3,124,550	8 436,285	21,474,180
19	33,649	134,596	480,700	1,562,275	4,686 824	13,123,110	34 597,290
20	42,504	177,100	657,800	2,220,075	6 906,900	20 029,010	54 677 300
21	53,130	230,230	888,030	3,108,105	10,015,005	30,045,015	84,672 315
22	65,780	296,010	1,184,040	4,292,145	14,307,150	44 352,165	129 024 480
23	80,730	376,740	1,560,780	5,852,925	20,160,075	64 512,240	193 536 720
24	98,280	475,020	2,035,800	7,888,725	28,048,800	92,561,040	286 097 760
25	118,755	593,775	2,629,575	10,518,300	38,567,100	131,128 140	417,225,900

TABLE II.

Terms.	6th Order.	7th Order.	8th Order.	9th Order.	10th Order.	11th Order.	12th Order.
26	142,506	736,281	3,365,856	13,884,156	52,451,256	183,579,396	600,805,296
27	169,911	906,192	4,272,048	18,156,204	70,607,460	254,186,856	854,992,152
28	201,376	1,107,568	5,379,616	23,535,820	94,143,280	348,330,136	1,203,322,288
29	237,336	1,344,904	6,724,520	30,260,340	124,403,620	472,733,756	1,676,056,044
30	278,256	1,623,160	8,347,680	38,608,020	163,011,640	635,745,396	2,311,801,440
31	324,632	1,947,792	10,295,472	48,903,492	211,915,132	847,660,528	3,159,461,968
32	376,992	2,324,784	12,620,256	61,523,748	273,438,880	1,121,099,408	4,280,561,376
33	435,897	2,760,681	15,380,937	76,904,685	350,343,565	1,471,442,973	5,752,004,349
34	501,942	3,262,623	18,643,560	95,548,245	445,891,810	1,917,334,783	7,669,339,132
35	575,757	3,838,380	22,481,940	118,030,185	563,921,995	2,481,256,778	10,150,595,910
36	658,008	4,496,388	26,978,328	145,008,513	708,930,508	3,190,187,286	13,340,783,196
37	749,398	5,245,786	32,224,114	177,232,627	886,163,135	4,076,350,421	17,417,133,617
38	850,668	6,096,454	38,320,568	215,553,195	1,101,716,330	5,178,066,751	22,595,200,368
39	962,598	7,059,052	45,379,620	260,932,815	1,362,649,145	6,540,715,896	29,135,916,264
40	1,086,008	8,145,060	53,524,680	314,457,495	1,677,106,640	8,217,822,536	37,353,738,800
41	1,221,759	9,366,819	62,891,499	377,348,994	2,054,455,634	10,272,278,170	47,626,016,970
42	1,370,754	10,737,573	73,629,072	450,978,066	2,505,433,700	12,777,711,870	60,403,728,840
43	1,533,939	12,271,512	85,900,584	536,878,650	3,042,312,350	15,820,024,220	76,223,753,060
44	1,712,304	13,983,816	99,884,400	636,763,050	3,679,075,400	19,499,099,620	95,722,852,680
45	1,906,884	15,890,700	115,775,100	752,538,150	4,431,613,550	23,930,713,170	119,653,565,850
46	2,118,760	18,009,460	133,784,560	886,322,710	5,317,936,260	29,248,649,430	148,902,215,280
47	2,349,060	20,358,520	154,143,080	1,040,465,790	6,358,402,050	35,607,051,480	184,509,266,760
48	2,598,960	22,957,480	177,100,560	1,217,566,350	7,575,968,400	43,183,019,880	227,692,286,640
49	2,869,685	25,827,165	202,927,725	1,420,494,075	8,996,462,475	52,179,482,355	279,871,768,995
50	3,162,510	28,989,675	231,917,400	1,652,411,475	10,648,873,950	62,828,356,305	342,700,125,300

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